LÖSNINGSFÖRSLAG TMA976, TENTA 3: 2021/22

1. Ansätt $y = x \cdot u \rightsquigarrow y' = u + x \cdot u' \rightsquigarrow y'' = 2u' + x \cdot u'' \rightsquigarrow y'' + y = x \cdot \underbrace{(u'' + u)}_{=0} + \underbrace{2u'}_{=\cos x} = \cos x$

$\rightsquigarrow u(x) = \frac{1}{2}\sin x \rightsquigarrow \boxed{y(x) = \frac{x}{2} \cdot \sin x}$ OBS: $y(0) = y(\pi) = 0$.

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2. • $y_0 = 0 \rightsquigarrow y(x) = 0 \ \forall x$ uppfyller initialvärdesproblemet.

• $y_0 \in (0, \pi/2)$: $y' \cdot \tan y = x \cdot \cos(x^2) \rightsquigarrow -\ln|\cos y(x)| = C + \frac{1}{2}\sin(x^2)$

$-\ln(\underbrace{\cos y_0}_{>0}) = C \rightsquigarrow \cos y(x) = e^{-C} \cdot e^{-\frac{1}{2}\sin(x^2)} = \underbrace{\underbrace{\cos y_0}_{\in(0,1)} \cdot \underbrace{e^{-\frac{1}{2}\sin(x^2)}}_{\leq 1 \text{ om } 0 \leq x \leq \sqrt{\pi}}}_{\in(0,1)}$

arccos def!

$\Rightarrow \boxed{y(x) = \arccos\left(\cos(y_0) \cdot e^{-\frac{1}{2}\sin(x^2)}\right)} \quad y(0) = y_0 \in (0, \pi/2)$.

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3. • $p < 1$: $S(p,q,r) \geq \sum_{n = n_{p,q,r}}^{\infty} \frac{1}{n} = \infty \longrightarrow S(p,q,r)$ divergerer $\forall q, r \geq 0$.

tillräckligt stort

• $p > 1$: $S(p,q,r) \leq \sum_{n=700}^{\infty} \frac{1}{n^p} < \infty$

• $p = 1$: Integraltest $S(1,q,r) < \infty \Leftrightarrow \int_{7000}^{\infty} \frac{dx}{x(\ln x)^q (\ln \ln x)^r} = \int_{\ln(7000)}^{\infty} \frac{dx}{x^q (\ln x)^r}$

$$\rightsquigarrow \boxed{S(p,q,r) < \infty \Leftrightarrow \begin{array}{l} p > 1, \ \forall q, r \geq 0 \\ p = 1, \ q > 1, \ \forall r \geq 0 \\ p = q = 1, \ r > 1 \end{array}}$$

• om $q > 1$ – konv $\forall r \geq 0$
om $q < 1$ – div $\forall r \geq 0$
• om $q = 1$:
$\int_{\ln(\ln 7000)}^{\infty} \frac{dx}{x^r} < \infty \Leftrightarrow r > 1$

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4. $\ln(1+u) = u + u^2/2 + O(u^3) \rightsquigarrow \ln(1+x^4) = x^4 + \frac{1}{2}x^8 + O(x^{12})$

$x^3 \cdot \sin(x) = x^3(x - x^3/3! + x^5/5! + O(x^7)) = x^4 - x^6/3! + x^8/5! + O(x^{10})$

$x^2 \cdot \cos(x^2) = x^2(x^2/2 - x^4/4! + x^6/6! + O(x^8)) = x^4/2 - x^6/4! + x^8/6! + O(x^{10})$

$\rightsquigarrow 2 \cdot \ln(1+x^4) - a \cdot x^3 \cdot \sin(x) - b \cdot x^2 \cdot \cos(x^2) = 2x^4 + \frac{1}{2}x^8 + O(x^{12})$

$- a\left(x^4 - x^6/3! + x^8/5!\right) - b \cdot \left(x^4/2 - x^6/4! + x^8/6!\right) + O(x^{10})$

$= \underbrace{(2 - a - b/2)}_{=0} x^4 + \underbrace{(a/3! + b/4!)}_{=0} x^6 + \left(\frac{1}{2} - \frac{a}{5!} - \frac{b}{6!}\right) x^8 + O(x^{10}) \Rightarrow \begin{cases} a + b/2 = 2 \\ a/3! + b/4! = 0 \end{cases}$

• $x^2 \cdot \tan^2(x) \cdot (e^{x^2} - 1) \cdot (\cos x - 1) = x^2 \cdot \underbrace{(x + O(x^3))^2}_{= x^2 + O(x^4)} \cdot (x^2 + O(x^4)) \cdot (-x^2/2 + O(x^4))$

$= -x^8/2 + O(x^{10})$

$b = -4a$

$\rightsquigarrow a - 2a = 2$

$\Rightarrow a = -2, \ b = 8$

$$\rightsquigarrow \boxed{I = -2\left(\frac{1}{2} - \frac{a}{5!} - \frac{b}{6!}\right) = -2\left(\frac{1}{2} + \frac{2}{5!} - \frac{8}{6!}\right)}$$

5. Konv.radie $R = \dfrac{1}{\lim_k |2^k|^{1/2k}} = 1 \rightsquigarrow$ Absolutkonv. om $|x|<1$, Div. $|x|>1$.

$x = \pm 1$: $\sum_{k=1}^{\infty} 2^k = \infty$ (div!)

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6. a) $u_k(\theta) = \dfrac{s_k(\theta)}{k^2}$ uppfyller $\underbrace{\max_\theta |u_k(\theta)|}_{=: M_k} \leq \dfrac{1}{k^2} \rightsquigarrow \sum_1^\infty M_k < \infty$

Weierstrass majorantTest: $\sum_1^\infty u_k$ konvergerar likformigt på $\mathbb{R}$ $\rightsquigarrow$ u_k kont. $\forall k$ $\rightsquigarrow \sum_1^\infty u_k$ kontinuerlig.

(Likf. konv. bevarar kont.)

b) Fix $\theta \in (0,\pi)$. Antag att $\lim_k \sin(k\theta) = 0$.

$k\theta = n_k\pi + \delta_k$, $n_k \geq 0$, $\delta_k \in [-1/2, 1/2)$, $\delta_k \to 0$.

$\underbrace{\theta}_{\text{oberoende av } k} = (k+1)\theta - k\cdot\theta = \underbrace{(n_{k+1}-n_k)}_{\text{e heltal}}\theta + \underbrace{\delta_{k+1}-\delta_k}_{\to 0} \iff \underbrace{(1-n_{k+1}+n_k)}_{\in \mathbb{Z}}\theta = \delta_{k+1}-\delta_k \to 0$

$\theta \neq 0$: Enda möjlighet är att $\delta_{k+1} = \delta_k$, $n_{k+1} = n_k + 1$ $\forall k$

$\rightsquigarrow k\theta = k\pi$ $\forall k$ tillr. stora.

Men $\theta \in (0,\pi/2)$ – omöjligt!

Motsägelse!

c) $\sin(k\theta) = \dfrac{1}{2i}\left(e^{ik\theta} - e^{-ik\theta}\right) = \operatorname{Im}(e^{ik\theta})$

$$\sum_{k=1}^{n} \sin(k\theta) = \sum_{k=0}^{n} \sin(k\theta) = \operatorname{Im}\left(\sum_{k=0}^{n} e^{ik\theta}\right)$$

$$= \operatorname{Im}\left(\frac{e^{i(n+1)\theta}-1}{e^{i\theta}-1}\right) = \operatorname{Im}\left(\frac{(e^{i(n+1)\theta}-1)(e^{-i\theta}-1)}{|1-e^{i\theta}|^2}\right)$$

$$= \frac{1}{|1-e^{i\theta}|^2}\cdot \operatorname{Im}\left(e^{in\theta} - e^{i(n+1)\theta} - e^{-i\theta} + 1\right) = \frac{\sin(n\theta) + \sin(n+1)\theta + \sin\theta}{2-2\cos\theta}$$

$$= \frac{1}{2}\cdot\left(\frac{\sin(n\theta) + \sin(n+1)\theta + \sin(\theta)}{2\sin^2(\theta/2)}\right) = \frac{1}{4}\cdot\left(\frac{\sin(n\theta)+\sin(n+1)\theta+\sin\theta}{\sin^2(\theta/2)}\right) \leq \frac{3}{4\sin^2(\theta/2)}$$

$$\Rightarrow \sup_n |s_n(\theta)| \leq \frac{3}{4\sin^2(\theta/2)} \quad \forall \theta \in (0,\pi).$$

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d) $s_n(0) = 0$ $\forall n$ $\Rightarrow$ $g(0) = 0$

$$g(1/n) - g(0) = g(1/n) \geq s_n(1/n) = \frac{1}{4}\left(\frac{\sin(1) + \overbrace{\sin\left(1+\frac{1}{n}\right)}^{\geq 0} + \overbrace{\sin\left(\frac{1}{n}\right)}^{\geq 0}}{\sin^2(1/2n)}\right)$$

$$\geq \frac{1}{4}\cdot\frac{\sin(1)}{\sin^2(1/2n)} \to \infty, \quad n\to\infty.$$

$\rightarrow$ g ej kontinuerlig i $\theta = 0$.