Recall: V vector space, $(\cdot,\cdot)$ inner product

Basis of V $\hookrightarrow$ Norm $\|u\| = \sqrt{(u,u)}$

Ex: $V = \mathbb{R}^n$, $(u,v) = u \cdot v = u^T v = \sum_{j=1}^{n} u_j v_j$, $\|u\| = \sqrt{\sum_{j=1}^{n} u_j^2}$

$V = L^2(a,b)$, $(f,g)_{L^2} = \int_a^b f(x) g(x)\,dx$, $\|f\|_{L^2} = \sqrt{\int_a^b f(x)^2\,dx}$

Ex: Compute the L^2-norm of $f(x) = x^2$ on $[0,1]$.

$$\|f\|_{L^2} \underset{\text{Def}}{=} \sqrt{\int_0^1 f(x)^2\,dx} = \sqrt{\int_0^1 x^4\,dx} = \sqrt{\frac{x^5}{5}\Big|_0^1} = \frac{1}{\sqrt{5}} \;//$$

Def: Let $(\overset{\text{vect. space}}{V}, \overset{\text{inner prod.}}{(\cdot,\cdot)})$ be an inner product space.

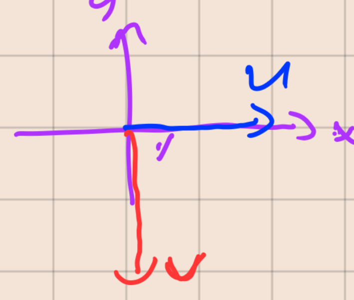
$u \in V$ and $v \in V$ are said orthogonal if

$$(u,v) = 0$$

Ex: $V = \mathbb{R}^2$. Are the vectors $u = \binom{2}{0}$ and $v = \binom{0}{-3}$ orthogonal?

$(u,v) = u \cdot v = (2 \cdot 0) + (0 \cdot (-3)) = 0$

$\Rightarrow u \perp v$ are orthogonal :-)



Ex: $V = \mathcal{P}^{(2)}(-1,1) = \{\text{polyn. defined on } (-1,1) \text{ of deg} \le 2\}$

$p(x) = x$, $q(x) = x^2$, $p \perp q$?

$$(p,q)_{L^2(-1,1)} \underset{\text{Def.}}{=} \int_{-1}^{1} p(x)q(x)\,dx = \int_{-1}^{1} x^3\,dx = \frac{x^4}{4}\Big|_{-1}^{1} = 0$$

$\Rightarrow$ $p \perp q$ orthogonal !

Th (Cauchy-Schwarz inequality (C-S))

Let $(V, (\cdot,\cdot))$ an inner product space and $u, v \in V$.

One has: $|(u,v)| \le \|u\| \cdot \|v\|$

where $\|u\| = \sqrt{(u,u)} \in \mathbb{R}$

Proof:

Let $\lambda \in \mathbb{R}$ and consider

$$0 \le \|u - \lambda v\|^2 \underset{\text{Def}}{=} (u - \lambda v, u - \lambda v) \overset{\text{linearity}}{=} (u, u - \lambda v)$$

$$- \lambda (v, u - \lambda v) \underset{\text{linearity}}{=} (u,u) - \lambda (u,v) - \lambda \underset{(u,v)}{\underset{\parallel}{(v,u)}} + \lambda^2 (v,v)$$

$= \|u\|^2 - \lambda(u,v) - \lambda(u,v) + \lambda^2 \|v\|^2$

Def $\|\cdot\|$, commutativity

$= \|u\|^2 - 2\lambda(u,v) + \lambda^2\|v\|^2$

c b a

This can be seen as a polyn. in λ of degree 2 and a non-negativ one!!

$\Rightarrow$ "$b^2 - 4ac \leq 0$" $\Rightarrow 4(u,v)^2 - 4\|u\|^2 \cdot \|v\|^2 \leq 0$

discremenent

$\Rightarrow (u,v)^2 \leq \|u\|^2 \cdot \|v\|^2$ square $\Rightarrow |(u,v)| \leq \|u\| \cdot \|v\|$

$\blacksquare$

Zeros $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(Theorem)

Th (Triangle inequality ($\triangle$))

Let $(V, (\cdot,\cdot))$ be an inner product space and $u, v \in V$. Then,

$$\|u+v\| \leq \|u\| + \|v\| \quad \left(\begin{array}{c} x,y \in \mathbb{R} \\ |x+y| \leq |x| + |y| \end{array} \right)$$

Proof:

Consider

$$\|u+v\|^2 \overset{\text{Def. norm}}{=} (u+v,u+v) \overset{\text{as above}}{=} (u,u)+2(u,v)+(v,v)$$

$$\underset{\text{Def norm}}{=} \|u\|^2 + 2(u,v) + \|v\|^2 \leq$$

$$\leq \|u\|^2 + 2\underbrace{|(u,v)|}_{?} + \|v\|^2 \quad \overset{\boxed{C-S}}{\leq}$$

$x \leq |x| \;\; \forall x \in \mathbb{R}$

$$\leq \|u\|^2 + 2\,\|u\|\cdot\|v\| + \|v\|^2 =$$

$$= (\|u\| + \|v\|)^2$$

$$\overset{\text{square root}}{\Longrightarrow} \quad \|u+v\| \leq \|u\| + \|v\| \quad :-)$$

$\blacksquare$

4) Function spaces:

Sol. to DE are functions $\Rightarrow$

need more function spaces!

Def: • The space of continuous fct on $[a,b]$ is defined by

$$\mathcal{C}^0([a,b]) = \mathcal{C}(a,b) = \mathcal{C}^{(0)}([a,b]) =$$
$$= \{ f : [a,b] \to \mathbb{R} : f \text{ continuous} \}$$

• The space

$$\mathcal{C}^1([a,b]) = \mathcal{C}^{(1)}([a,b]) = \{ f : [a,b] \to \mathbb{R} : f \text{ and } f' \text{ are cont.} \}$$
$$\shortparallel$$
$$C^1([a,b])$$

⋮

• The space

$$\mathcal{C}^{(k)}([a,b]) = \{ f : [a,b] \to \mathbb{R} : f, f', \dots, f^{(k)} \text{ cont.} \}$$

• With the norm

$$\|f\|_{C^0} = \|f\|_{C([a,b])} = \max_{a \le x \le b} |f(x)|$$

$$\|f\|_{C^1} = \|f\|_{C^1([a,b])} = \|f\|_{C([a,b])} + \|f'\|_{C([a,b])}$$

$$= \max_{a \le x \le b} \left(|f(x)| + |f'(x)| \right)$$

$\vdots$

<u>Rem</u>: We can define these spaces for $f:(a,b) \to \mathbb{R}$ or $f:(a,b] \to \mathbb{R}$ ... need some adaptations.

$(\max \longleftrightarrow \underset{\text{(supremum)}}{\sup})$

<u>Ex</u>: Compute the C^0 and C^1 norms of $f(x) = x^2 + e^x$ on $[0,2]$:

$$\|f\|_{C^0(0,2)} \overset{\text{Def norm}}{=} \max_{0 \le x \le 2} |\underbrace{x^2 + e^x}_{\geq 0}| =$$

$$= \max_{0 \le x \le 2} (x^2 + e^x) \underset{P}{=} 2^2 + e^2 = e^2 + 4$$

$x^2, e^x \nearrow$ on $[0, 2]$

$$\|f\|_{C^1(0,2)} \overset{\substack{\text{Def} \\ b}}{=} \|f\|_{C^0(0,2)} + \|f'\|_{C^0(0,2)} =$$

$$\underset{\substack{P \\ \text{above}}}{=} (4+e^2) + \max_{0 \le x \le 2} |\underbrace{2x + e^x}_{\ge 0 \text{ and } \nearrow \text{increase}}|$$

$$= (4+e^2) + (4+e^2) = 2e^2 + 8$$

<u>Def</u>: • Let $1 \le p < \infty$, define

$$L^p(a,b) = L_p(a,b) = \left\{ f : [a,b] \to \mathbb{R} : \|f\|_{L^p} < \infty \right\}$$

where $\|f\|_{L^p} = \left(\int_a^b |f(x)|^p dx \right)^{1/p}$ (if $p = 2$ $\to L^2$, seen before)

• For "$p = \infty$", define

$$L^\infty(a,b) = L_\infty(a,b) = \{ f : [a,b] \to \mathbb{R} : \|f\|_{L^\infty} < \infty \}$$

(continuous)

where $$\|f\|_{L^\infty(a,b)} = \max_{a \leq x \leq b} |f(x)|$$

Ex: Compute the L^1-norm of

$f(x) = 2x + \sin(x)$ for $x \in [0, \frac{\pi}{2}]$?

$$\|f\|_{L^1[0,\frac{\pi}{2}]} \underset{\text{Def}}{=} \int_0^{\pi/2} |\underbrace{2x + \sin(x)}_{\geq 0}| \, dx =$$

$$= \int_0^{\pi/2} (2x + \sin(x)) \, dx = x^2 - \cos(x) \Big|_0^{\pi/2} =$$

$$= \left(\frac{\pi^2}{4} - 0\right) - (0 - 1) = \frac{\pi^2}{4} + 1 \quad //$$

Chapter III: Polynomial approximation in 1D

Goal: Approximate (complicated) fct by polynomials

Application: approx. sol. to DE.

1) Some more spaces:

Recall: $\mathcal{P}^{(q)}(a,b) = \{$polyn. defined $[a,b]$ and of deg $\leq q\}$.

- The space of trigonometric polynomials on $[0,L]$ is

$$T^N(0,L) = \text{span}\Big(1, \cos\left(\tfrac{2\pi}{L}x\right), \sin\left(\tfrac{2\pi}{L}x\right),$$
$$\cos\left(\tfrac{2\pi}{L}\cdot 2\cdot x\right), \sin\left(\tfrac{2\pi}{L}\cdot 2\cdot x\right), \ldots, \cos\left(\tfrac{2\pi N}{L}x\right),$$

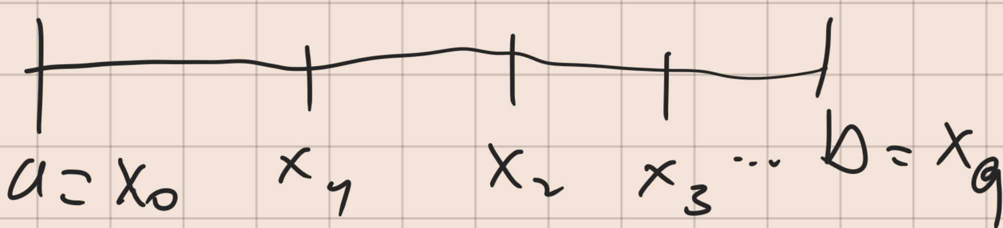
$\sin\left(\frac{2\pi}{L}Nx\right)\Big\} = \Big\{ f :$

$$f(x) = \sum_{n=0}^{N} \left(a_n \cdot \cos\left(\frac{2\pi}{L} n \cdot x\right) + b_n \sin\left(\frac{2\pi}{L} n x\right)\right)\Big\}$$

(use later on)

coeff. (real numb.)

- Consider $[a,b]$ and a grid of $(q+1)$ distinct points



Define Lagrange polynomials by

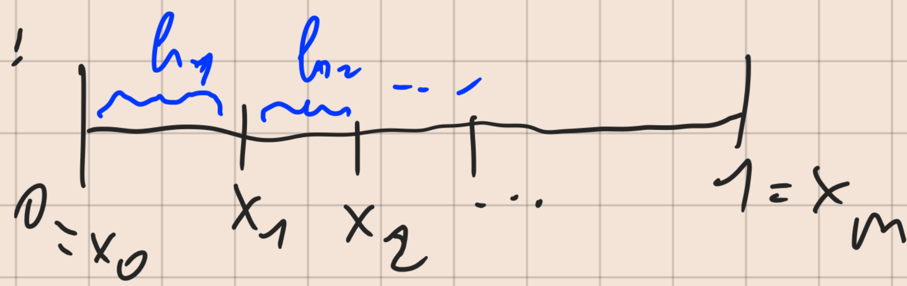
$$\lambda_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{q} \frac{x - x_j}{x_i - x_j} \quad \text{for } i = 0, 1, \dots, q$$

$$\left(\lambda_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases} \right)$$

$$\mathcal{P}^{(q)}(a,b) = \operatorname{span}\left(\lambda_0(x), \lambda_1(x), \dots, \lambda_q(x) \right)$$

no proof.

- Consider $[0,1]$ and $m+1$ subintervals $(x_{j-1}, x_j]$:



Denote $h_j = x_j - x_{j-1}$ for $j = 1, 2, \dots, m$

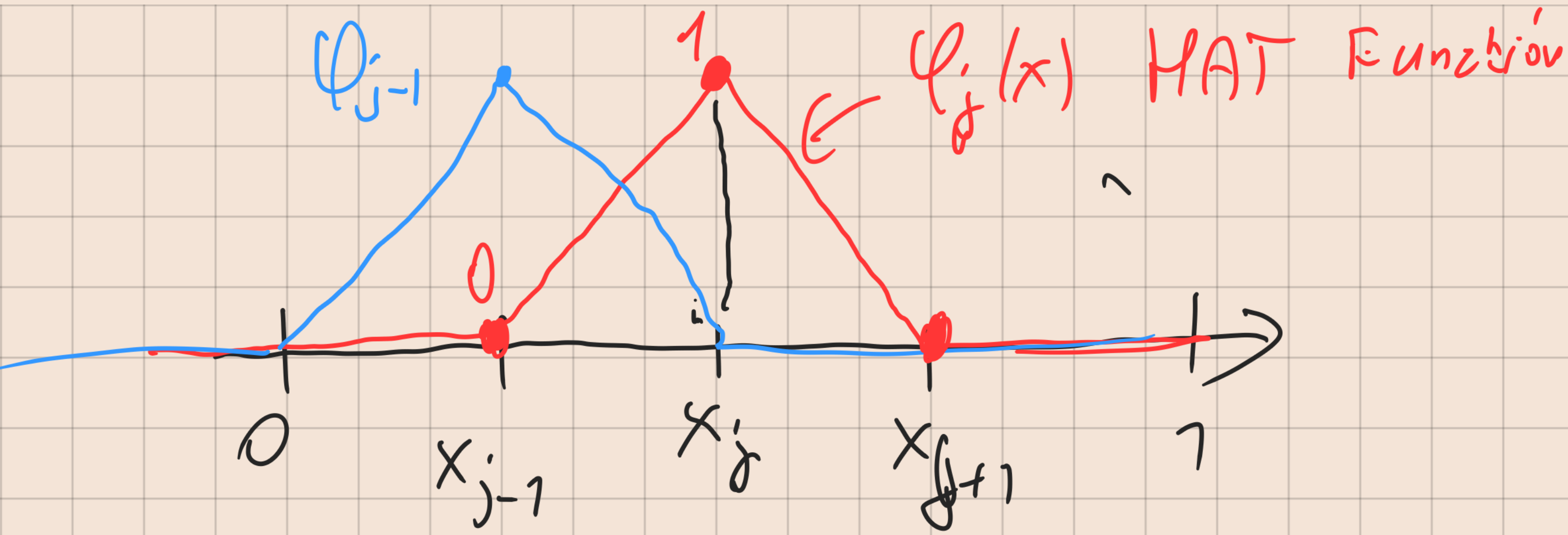
This gives us a partition of $[0,1]$:

$T_h : 0 = x_0 < x_1 < x_2 \cdots < x_m = 1$,

where $h_j = x_j - x_{j-1}$

Define hat function:

$$\varphi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{h_j} & \text{for } x_{j-1} \le x \le x_j \\[2ex] \dfrac{x - x_{j+1}}{-h_{j+1}} & \text{for } x_j < x \le x_{j+1} \\[2ex] 0 & \text{else} \end{cases}$$



φ_{j-1}
1
$\varphi_j(x)$ HAT Funktion
0
0
x_{j-1}
x_j
x_{j+1}
1