Recall:

- Heat equation (PDE)

$$(H)\begin{cases} u_t(x,t) - u_{xx}(x,t) = f(x,t) & 0<x<1,\ 0<t\le T \\ u(0,t) = 0 = u(1,t) & 0<t\le T \\ u(x,0) = u_0(x) & 0<x<1 \end{cases}$$

- (VF) Find $u(\cdot,t) \in V^0$ s.t. for all $0<t\le T$,

$$(u_t(\cdot,t), v)_{L^2(0,1)} + (u_x(\cdot,t), v_x)_{L^2} = (f(\cdot,t), v)_{L^2} \quad \forall v \in V^0$$

$$u(x,0) = u_0(x)$$

- (FE) Find $U(\cdot,t) \in V_h^0$ s.t. for all $0<t\le T$,

$$(U_t(\cdot,t), \chi)_{L^2} + (U_x(\cdot,t), \chi_x)_{L^2} = (f(\cdot,t), \chi)_{L^2} \quad \forall \chi \in V_h^0$$

$$U(x,0) = \Pi_h u_0(x) \quad \text{(cont pw. linear interpolant of } u_0\text{)}$$

(see Chapter 4)

(iii) From (FE), we get a linear system of ODE by choosing $\chi(x) = \varphi_i(x)$ for $i = 1,2,\dots,m$ and by writing $U(x,t) = \sum_{j=1}^{m} \xi_j(t)\cdot\varphi_j(x)$,

$\mathrm{span}(\varphi_1,\dots,\varphi_m) = V_h^0$ — depend on t

where φ_j are hat functions. We insert this in (FE) to get:

$$\underbrace{\left(\sum_{j=1}^{m} \dot{\xi}_j(t)\varphi_j, \varphi_i\right)_{L^2}}_{\sum_{j=1}^{m} \dot{\xi}_j(t)\underbrace{(\varphi_j,\varphi_i)_{L^2}}_{m_{ij}}} + \underbrace{\left(\sum_{j=1}^{m} \xi_j(t)\varphi_j', \varphi_i'\right)_{L^2}}_{\sum_{j=1}^{m} \xi_j(t)\underbrace{(\varphi_j',\varphi_i')}_{s_{ij}}{}_{L^2}} = \overbrace{(f(\cdot,t),\varphi_i)_{L^2}}^{F_i(t)}$$

for $i=1,\dots,m$.

(linearity inner prod)

At the end, we get the syst. of linear ODE

$$M\dot{\xi}(t) + S\xi(t) = F(t),$$

where M $(m\times m)$ mass matrix

S $(m\times m)$ stiffness matrix

$F(t)$ $(m\times 1)$ load vector

$\xi(t)$ $(m\times 1)$ unknown vector

We get the initial condition $\xi(0)$ as follow:

$$U(x,0) = \underset{\text{(FE)}}{=} \Pi_h u_0(x) \underset{\text{Def interpolant}}{\overset{\text{Def } U}{\iff}} \underline{\sum_{j=1}^{m} \zeta_j(0)\cdot\varphi_j(x)} \overset{!}{=} \underbrace{\sum_{j=1}^{m} u_0(x_j)\varphi_j(x)}$$

$$\underline{\sum_{j=1}^{m} u_0(x_j)\varphi_j(x)}$$

Hence, $\zeta_j(0) = u_0(x_j)$ for $j = 1, \dots, m. \Rightarrow$

$$\zeta(0) = \begin{pmatrix} u_0(x_1) \\ u_0(x_2) \\ \vdots \\ u_0(x_m) \end{pmatrix}$$

(iv) In general the above ODE is difficult to solve $\leadsto$ we use (f.ex) backward Euler scheme to find a numerical approximation. We define time grid $t_n = n\cdot k$, for $n = 0, 1, \dots, N$, and time step $k = \frac{T}{N}$.

$$M\left(\frac{\xi^{(n+1)}-\xi^{(n)}}{k}\right)+S'\xi^{(n+1)}=F(t_{n+1})$$

$\Leftrightarrow$

$$(M+kS')\xi^{(n+1)}=M\xi^{(n)}+kF(t_{n+1}) \quad (*)$$

$$\xi^{(0)}=\xi(0)=\begin{pmatrix} u_0(x_1) \\ u_0(x_2) \\ \vdots \\ u_0(x_m) \end{pmatrix}$$

This gives us approx. $\xi^{(n)} \approx \xi(t_n)$ and thus the approx

$$\underbrace{U(x,t_n)}_{\approx\, u(x,t_n)} \approx \sum_{j=1}^{m} \xi_j^{(n)} \varphi_j(x)$$

Rem: See $(*)$ (Euler)

- $F(t)$ is an integral, one

may use a Quadrature Formula (midpoint rule, etc.)

- One needs to solve a linear syst. of eq. to find $\xi^{(n+1)}$, for all $n = 0, 1, 2, \ldots, N-1$.

3) The wave equation in 1d:

Simple model to describe a vibrating violin string $(0 \le x \le 1,\ 0 < t \le T)$:

Wave eq. (W) $\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = f(x,t) \\ u(0,t) = 0 = u(1,t) \\ u(x,0) = u_0(x) \\ u_t(x,0) = v_0(x), \end{cases}$

where u_0, v_0, f are given fct.

Th: For the wave eq. (W) with $f \equiv 0$,

one has conservation of energy:

$$\underbrace{\frac{1}{2}\|u_t(\cdot,t)\|_{L^2}^2 + \frac{1}{2}\|u_x(\cdot,t)\|_{L^2}^2}_{\text{energy at time } t} = \underbrace{\frac{1}{2}\|v_0\|_{L^2}^2 + \frac{1}{2}\|u_0'\|_{L^2}^2}_{\substack{\text{initial energy} \\ \text{(constant)}}}$$

Kinetic energy   potential

$\forall t$

Proof:

Idea: Multiply the DE with $u_t(x,t)$:

$$u_{tt}(x,t)\,u_t(x,t) - u_{xx}(x,t)\,u_t(x,t) = 0$$

Then integrate over $x \in [0,1]$, int. by parts:

$$\underbrace{\int_0^1 u_{tt}(x,t)u_t(x,t)\,dx}_{\frac{1}{2}\frac{d}{dt}\left(u_t(x,t)^2\right)} - \underbrace{\int_0^1 u_{xx}(x,t)u_t(x,t)\,dx}_{+\int_0^1 u_x(x,t)\,u_{tx}(x,t)\,dx + u_x(x,t)u_t(x,t)\Big|_{x=0}^{1}} = 0$$

$$\underbrace{\frac{1}{2}\frac{d}{dt}\left(u_t(x,t)^2\right)}_{\frac{1}{2}2u_tu_{tt}}$$

(Chain rule)

$$\underbrace{u_x(x,t)\,u_{tx}(x,t)}_{\frac{1}{2}\frac{d}{dt}\left(u_x^2(x,t)\right)}$$

$u_x(x,t)u_t(x,t)\Big|_{x=0}^{1} = 0$ due BC

This gives us

$$\frac{1}{2}\frac{d}{dt}\left\{\underbrace{\int_0^1 (u_t(x,t))^2\,dx}_{\|u_t(\cdot,t)\|_{L^2}^2} + \underbrace{\int_0^1 (u_x(x,t))^2\,dx}_{\|u_x(\cdot,t)\|_{L^2}^2}\right\} = 0$$

(Def L^2-norm)

$$\Longrightarrow \quad \frac{1}{2}\|u_t(\cdot,t)\|_{L^2}^2 + \frac{1}{2}\|u_x(\cdot,t)\|_{L^2}^2 \equiv \text{Const}$$

initial energy

$\frac{d}{dt}(\dots) = 0 \Rightarrow \dots \equiv$ const.

:-) ∎

<u>Rem:</u> Rewrite (W) as a system of DE by introducing a new variable $v = u_t$ (velocity)

$$u_t = v$$

$$v_t = u_{tt} \underset{\text{Def (W)}}{=} u_{xx} + f \quad \Leftrightarrow \quad \begin{pmatrix} u \\ v \end{pmatrix}_t = \underbrace{\begin{pmatrix} 0 & 1 \\ \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} u \\ v \end{pmatrix}}_{W} + \underbrace{\begin{pmatrix} 0 \\ f \end{pmatrix}}_{F}$$

$$\Leftrightarrow \quad W_t = AW + F.$$

4) Discretisation of wave in 1d:

Apply FEM (in space, x) and Crank-Nicolson (in time, t) to get a numerical approx. of sol. to (W).

(i) The variational formulation reads

(VF) Find $u(\cdot,t) \in V^0$, s.t. for $0 < t \leq T$, one has

$$(u_{tt}(\cdot,t), v)_{L^2} + (u_x(\cdot,t), v_x)_{L^2} = (f(\cdot,t), v)_{L^2} \quad \forall v \in V^0$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x).$$

same space as for heat

(ii) The FE problem is obtained as for the heat eq.

and one gets the following problem:

(FE) Find $U(\cdot,t) \in V_h^0$ s.t. for $0 < t \leq T$, one has

$$(U_{tt}(\cdot,t), \chi)_{L^2} + (U_x(\cdot,t), \chi_x)_{L^2} = (f(\cdot,t), \chi)_{L^2} \quad \forall \chi \in V_h^0$$

$$U(x,0) = \Pi_h u_0(x), \quad U_t(x,0) = \Pi_h v_0(x)$$

same space as for heat eq.

(iii) To get a syst. of linear ODE from (FE),

we choose $\chi = \varphi_i$, and write

$$U(x,t) = \sum_{j=1}^{m} \underbrace{\xi_j(t)}_{\text{unknown}} \underbrace{\varphi_j(x)}_{\text{hat fct}}$$

insert the above into (FE) and get the following ODE:

(ODE) $\quad M\ddot{\xi}(t) + S\xi(t) = F(t)$, where

$M \rightarrow$ mass matrix

$S \rightarrow$ stiffness matrix

$F(t) \rightarrow$ load vector

$\xi(t) \rightarrow$ unknown vector

$\left(\ddot{\xi}(t) = \frac{d^2}{dt^2}\xi(t) \right)$

(IV) In order to be able to use CN (= Crank-Nicolson) we need to write (ODE) as a system of first order DE. Introduce new variable $\dot{\xi} = \eta$ and get

$$\begin{cases} M\dot{\xi}(t) = M\eta(t) & \text{(def of } \eta\text{)} \\ M\dot{\eta}(t) + S\xi(t) = F(t) & \text{(def ODE)} \end{cases}$$

Now we apply CN to get

$$M\left(\frac{\zeta^{(n+1)}-\zeta^{(n)}}{k}\right)=M\,\frac{1}{2}\left(\eta^{(n+1)}+\eta^{(n)}\right)$$

$$M\left(\frac{\eta^{(n+1)}-\eta^{(n)}}{k}\right)+S\,\frac{1}{2}\left(\zeta^{(n+1)}+\zeta^{(n)}\right)=\frac{1}{2}\left(F(t_n)+F(t_{n+1})\right)$$

or

$$\begin{pmatrix} M & -\frac{k}{2}M \\ \frac{k}{2}S & M \end{pmatrix}\begin{pmatrix} \zeta^{(n+1)} \\ \eta^{(n+1)} \end{pmatrix}=\begin{pmatrix} M & \frac{k}{2}M \\ -\frac{k}{2}S & M \end{pmatrix}\begin{pmatrix} \zeta^{(n)} \\ \eta^{(n)} \end{pmatrix}+$$

$$+\begin{pmatrix} 0 \\ \frac{k}{2}\left(F(t_n)+F(t_{n+1})\right) \end{pmatrix}$$

Th: For $f\equiv 0$, the energy is constant for CN, but not for Euler schemes!!