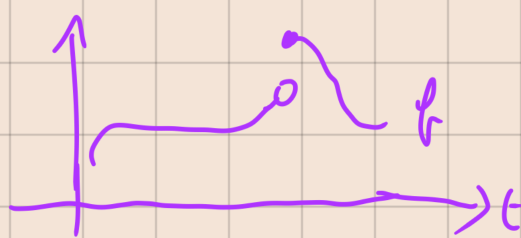
**Recall:** $f$ pwc and of exponential order $a$ has the Laplace transform (LT)

$$F(s) = \mathcal{L}\{f\}(s) = \int_0^\infty f(t) e^{-st} \, dt \quad \text{for } s > a$$




pwc


exp

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**Th.** (Shifting 1)

If $f$ has the LT $F(s) = \mathcal{L}\{f\}(s)$ and $c \in \mathbb{R}$, then

$$\mathcal{L}\{e^{ct} f(t)\}(s) = F(s-c) \quad \text{for } s > c.$$

**Proof.**

$$\mathcal{L}\{e^{ct} f(t)\}(s) \underset{\text{Def LT}}{=} \int_0^\infty e^{ct} f(t) e^{-st} \, dt = \int_0^\infty f(t) e^{-(s-c)t} \, dt =$$

$$\underset{\text{Def LT}}{=} F(s-c)$$

Def LT $F(u) = \int_0^\infty f(t) e^{-ut} \, dt$

$\blacksquare$

Ex: Compute $\mathcal{L}\{3e^{-2t}\cos(6t)\}(s)$?

$$\mathcal{L}\{3e^{-2t}\cos(6t)\}(s) \underset{\text{linearity LT}}{=} 3\,\mathcal{L}\{e^{-2t}\cos(6t)\}(s) =$$

$c=-2$ $\quad f(t)=\cos(6t)$

$$F(s)=\frac{s}{s^2+6^2}$$

(Table)

Theorem

$$= 3F(s-(-2)) = 3F(s+2) = 3\,\frac{s+2}{(s+2)^2+36}$$

Th (Shifting 2)

If $f(t-T)$ is zero for $t<T$, then

$$\mathcal{L}\{f(t-T)\}(s) = e^{-sT}F(s) \quad \text{for } s>0.$$

Proof:

$$\mathcal{L}\{f(t-T)\}(s) \underset{\text{Def LT}}{=} \int_0^\infty f(t-T)e^{-st}\,dt =$$

$$\underset{\text{Def of } f}{=} \int_T^\infty f(t-T)e^{-st}\,dt \underset{\tau=t-T,\ d\tau=dt}{=} \int_0^\infty f(\tau)\cdot e^{-s(\tau+T)}\,d\tau =$$

$$= e^{-sT} \underbrace{\int_0^\infty f(\tau) e^{-s\tau} d\tau}_{F(s) = \mathcal{L}\{f\}(s) \text{ by Def of LT}} = e^{-sT} F(s)$$

$$F(s) = \int_0^\infty f(t) e^{-st} dt$$

$\blacksquare$

Rem: If $f(t-T)$ is not zero up to $T$, simply use Heaviside fct:

$$\mathcal{L}\{f(t-T)\Theta(t-T)\}(s) = e^{-sT} F(s)$$

Th: (Derivative of LT)

Let $F$ be the LT of $f$. If $F'$ $\exists$, then

$$\mathcal{L}\{t f(t)\}(s) = -F'(s)$$

Similarly, one has

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n F^{(n)}(s) \quad \ldots$$

**Proof:**

Def LT

We have $\frac{d}{ds} F(s) = \frac{d}{ds} \int_0^\infty f(t) e^{-st} dt =$

$$= \int_0^\infty f(t) \frac{d}{ds} e^{-st} dt = \int_0^\infty f(t)(-t) e^{-st} dt =$$

$$= -\int_0^\infty (t f(t)) e^{-st} dt = -\mathcal{L}\{t f(t)\}(s)$$

$$G(s) = \mathcal{L}\{g\}(s) = \int_0^\infty g(t) e^{-st} dt$$

$$g : [0, \infty) \to \mathbb{R}$$
$$t \longmapsto g(t)$$

$$G : \mathbb{R} \to \mathbb{R}$$
$$s \longmapsto G(s) = \int_0^\infty g(t) e^{-st} dt$$

Ex: Compute $\mathcal{L}\{t e^{2t}\}(s) = ?$

$f(t) = e^{2t}$

$F(s) = \mathcal{L}\{f\}(s) = \frac{1}{s-2}$

Table $(a=-2)$

Using the above result, we get

$$\mathcal{L}\{t e^{2t}\}(s) = -F'(s) = -\frac{d}{ds}\left(\frac{1}{s-2}\right) =$$

$$= \frac{1}{(s-2)^2} \,//$$

Th (Integral of LT)

Let $F(s)$ be the LT of $f$ and assume (+ $F$ is integrable) that $\lim_{t\to 0} \frac{1}{t} f(t) \; \exists$. Then,

$$\mathcal{L}\left\{\frac{1}{t} f(t)\right\}(s) = \int_s^{\infty} F(w)\, dw$$

Proof:

Let us start by using the def of LT:

$$\int_s^\infty F(w)\,dw \underset{\text{Def LT } F}{=} \int_s^\infty \int_0^\infty f(t)\, e^{-wt}\, dt\, dw \underset{dt\,dw \to dw\,dt}{=}$$

$$= \int_0^\infty \int_s^\infty f(t) e^{-wt}\, dw\, dt =$$

$$= \int_0^\infty f(t) \left( \frac{e^{-wt}}{(-t)} \Bigg|_{w=s}^{\infty} \right) dt =$$

$$= \int_0^\infty f(t) \left(0 + \frac{e^{-st}}{t}\right) dt = \int_0^\infty \left(\frac{f(t)}{t}\right) e^{-st}\, dt =$$

$$= \mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) \quad ;-)$$

■

Ex: Compute LT $\mathcal{L}\left\{\frac{\sin(t)}{t}\right\}(s)$?

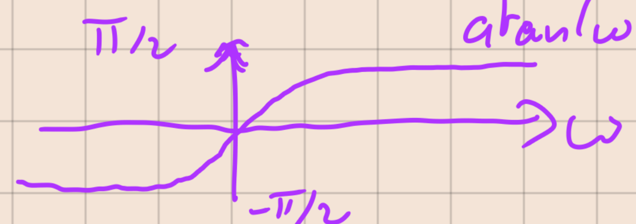
We can use the above result

with $f(t) = \sin(t)$ and $F(s) = \frac{1}{s^2+1}$

Observe that $\lim_{t\to 0} \frac{f(t)}{t} = \lim_{t\to 0} \frac{\sin(t)}{t} = 1$

Hospital

$$\Rightarrow \mathcal{L}\left\{\frac{\sin(t)}{t}\right\}(s) = \int_s^\infty \frac{1}{w^2+1}\,dw = \arctan(w)\Big|_{w=s}^{\infty} = \frac{\pi}{2} - \arctan(s)$$




**Th:** (LT of derivatives)

Let $F(s)$ be LT of $f$ and assume that $f'$ has a LT. Then, one has

$$\mathcal{L}\{f'(t)\}(s) = s\cdot F(s) - f(0)$$

(for $s > a$, $f'$ exp. order $a$)

**Proof:**

$$\mathcal{L}\{f'(t)\}(s) \overset{\text{Def LT}}{=} \int_0^\infty f'(t)e^{-st}\,dt \underset{\text{By part}}{=}$$

$$= f(t)e^{-st}\Big|_{t=0}^{\infty} - \int_0^\infty f(t)(-s)e^{-st}\,dt =$$

$$= 0 \quad -f(0) + s\underbrace{\int_0^\infty f(t)e^{-st}\,dt}_{F(s) \text{ by def of LT}} =$$

$$= -f(0) + sF(s)$$

■

<u>Rem</u>: Similarly

$$\mathcal{L}\{f^{(n)}(t)\}(s) = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0)$$

(under appropriate assumptions)

<u>Th</u>: (LT of integrals)

Let $F(s)$ denote LT of $f(t)$. If $f$ integrable, then

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\}(s) = \frac{F(s)}{s}, \quad (s>0)$$

Proof:

- Set $h(t) := \int_0^t f(\tau)\,d\tau$ and observe

$$h'(t) = f(t) \quad \text{and} \quad h(0) = \int_0^0 \dots = 0$$

- Use the previous result:

$$\underbrace{\mathcal{L}\{h'(t)\}(s)}_{= \mathcal{L}\{f(t)\}(s) = F(s)} = -\underbrace{h(0)}_{=0} + s\,\mathcal{L}\{h(t)\}(s) =$$

$$= 0 + s\,\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\}(s)$$

$$\Rightarrow \mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\}(s) = \frac{F(s)}{s}$$

$\blacksquare$

## 3) Inverse Laplace transforms:

**Def:** The inverse Laplace transform (ILT) of $F(s)$ is the (pwc and of expon. order) function $f(t)$ s.t. $\mathcal{L}\{f(t)\}(s) = F(s)$.

Notation: $\boxed{f(t) \equiv \mathcal{L}^{-1}\{F(s)\}(t)}$

**Rem:**
- Use a table of LT to find ILT.
- $\mathcal{L}^{-1}$ is also linear:

$$\mathcal{L}^{-1}\{aF + bG\}(t) = a\mathcal{L}^{-1}\{F\}(t) + b\mathcal{L}^{-1}\{G\}(t).$$

**Ex:** Find ILT of $F(s) = \frac{2}{s} - \frac{3}{s^2} + \frac{4s}{s^2+1}$

We need to compute

$$f(t) = \mathcal{L}^{-1}\{F(s)\}(t) = \mathcal{L}^{-1}\left\{\frac{2}{s} - \frac{3}{s^2} + \frac{4s}{s^2+1}\right\}(t)$$

$$= 2 \cdot \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}(t) - 3 \cdot \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}(t) +$$

linearity

$$+ 4\, \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}(t) =$$

Table

$$= 2 \cdot \Theta(t) - 3t + 4\cos(t)$$