TMA683 - HT21 ⑴

Chapter I: Introduction and motivation

See intro.pdf

Terminology:

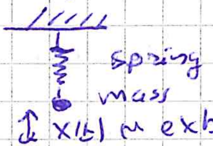
- A differential equation (DE) is an equation that relates an unknown fct (or more) and its derivatives.
- An ordinary differential equation (ODE) is a DE, where the unknown fct depends only on one variable (say $y(x)$)
- A partial differential equation (PDE) is a DE, when the unknown fct depends on 2 or more variables (say $u(x,y)$)

Ex: (ODE)

- Malthusian growth model for bacteria $\frac{d}{dt}P(t) = \lambda \cdot P(t)$ — λ: given parameter; $P(t)$: # of bacteria at time t

  Obs: $\dot{P}(t) = \frac{d}{dt}P(t)$ (notation)

- Mass-spring system



j + Hook

Newton's law $m \cdot \frac{d^2}{dt^2} x(t) + k \cdot x(t) = 0$ — m: mass; $\frac{d^2}{dt^2}x(t)$: accel.; $k \cdot x(t)$: force (k ↔ stiffness const.)

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- $u'(x) + 5u(x) + 22\cos(x) = 0$

To determine a unique sol. to a DE, one needs to specify additional conditions. For instance:

Ex: (IVP)

Add an initial condition ⇝ this gives us an initial value prob (IVP)

$$\begin{cases} \frac{d}{dt}P(t) = \lambda P(t) & \text{(DE)} \\ P(0) = P_0 & \text{(IV/IC)} \leftarrow \text{initial population of bacteria} \end{cases}$$

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ex: $\begin{cases} \frac{d}{dt}P(t) = 2 \cdot P(t) \\ P(0) = 1 \end{cases}$ has solution $P(t) = 1 \cdot e^{2t}$

Ex: (BVP)

Add boundary conditions ⇝ gives a boundary value problem (BVP)

$$\begin{cases} -u''(x) = \cos(x) & \text{for } x \in (0,1) \quad \text{(DE)} \\ u(0) = 0, \ u(1) = 5 & \quad \text{(BC)} \end{cases}$$

Let us now introduce the <u>Laplace operator</u>

In 2D (for 2 variables x, y): $\Delta u(x,y) = \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = u_{xx} + u_{yy}$

In 3D (" 3 " x, y, z): $\Delta u(x,y,z) = u_{xx} + u_{yy} + u_{zz}$, etc.

Notation in physics $\nabla^2 u = \Delta u$

<u>Ex</u>: Consider $u(x,y) = 5x^2y^4$, compute $\Delta u(x,y)$:

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$$\left.\begin{array}{l} u_x = 10xy^4 \rightsquigarrow u_{xx} = 10y^4 \\ u_y = 20x^2y^3 \rightsquigarrow u_{yy} = 60x^2y^2 \end{array}\right\} \Rightarrow \Delta u = u_{xx} + u_{yy} = 10y^4 + 60x^2y^2$$

<u>Ex</u>: (PDE)

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- <u>Laplace equation</u>: $\Delta u = 0$ (in 1D, 2D, 3D, ..)
- <u>Heat equation</u> (1D) $u_t - u_{xx} = f$ $\left(u_t(x,t) - u_{xx}(x,t) = f(x,t,u) \right)$

  describes evolution of temperature in a thin wire
- <u>Wave equation</u> (1D) $u_{tt} - u_{xx} = g$ $\left(u_{tt}(x,t) - u_{xx}(x,t) = g(x,t,u) \right)$

  describes motion of guitar string

Again, one needs to add conditions to find a unique sol. to a PDE

<u>Ex</u>: Consider heat eq. on $[0,1]$:

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = 0 & \text{for } x \in (0,1) \text{ and } t \in (0,T] \quad \text{(DE)} \\ u(0,t) = 0, \ u(1,t) = \sin(t) & \text{for } t \in (0,T] \quad \text{(BC)} \\ u(x,0) = 3x & \text{for } x \in (0,1) \quad \text{(IC)} \end{cases}$$

<u>Chapter II</u>: Mathematical tools

<u>Goal</u>: Introduce some (abstract) spaces, where we will live and work.

1) <u>Vector spaces</u>:

<u>Rem</u>: If vectors in $\mathbb{R}^n$ OK for you, below is the same!

<u>Def</u>: A set V of "vectors" or fcts is called a <u>Vector space</u> or linear space if $\forall u, v \in V$ and $\forall \alpha, \beta \in \mathbb{R}$, one has:

axioms (i) $u+(v+w) = (u+v)+w$ (associativity) ②

(ii) $u+v = v+u$ (commutativity)

(iii) $\exists\, 0 \in V$ s.t. $v+0 = v$ (~~identity~~ zero vector)

(iv) $\forall v \in V$, $\exists$ inverse $(-v) \in V$ s.t. $v+(-v) = (-v)+v = 0$ (inverse)

(v) $\alpha(\beta v) = (\alpha\beta) v$ (compatibility)

(vi) $\exists\, 1 \in \mathbb{R}$ s.t. $1 \cdot v = v$ (identity) [ok since $\mathbb{R}$]

(vii) $\alpha(u+v) = \alpha u + \alpha v$

(viii) $(\alpha+\beta) v = \alpha v + \beta v$ (distributivity)

Ex: • $V = \mathbb{R}^2 = \{(x,y) : x,y \in \mathbb{R}\}$ with $(x_1,y_1)+(x_2,y_2) = (x_1+x_2, y_1+y_2)$

$\alpha \cdot (x,y) = (\alpha x, \alpha y) \quad \forall \alpha \in \mathbb{R}$

verifies (i) – (viii) and is a vector space/linear space

• $V = \mathbb{R}^3 = \{(x,y,z) : x,y,z \in \mathbb{R}\}$ + and • as above

• $V = \mathbb{R}^n$ for $n \in \mathbb{N}$.

Ex: How can we define the space of polynomials of degree ≤ 1 on $(0,1)$?

$V = \{ p(x)$ polynomial with $\deg(p(x)) \le 1$ and $x \in (0,1)\} =$

⑤ $= \{ax+b : a,b \in \mathbb{R}, 0<x<1\} = \mathcal{P}^{(1)}(0,1)$

$\rightsquigarrow$ this is a vector space / linear space with usual +, •

Ex: More generally, $\mathcal{P}^{(n)}(\mathbb{R})$ denotes the linear space of all polynomials of degree $\le n$ on $\mathbb{R}$.

Ex: $\mathcal{P}^{(3)}(\mathbb{R})$ contains all polyn. of the form

$a_0 \,/\, a_0+a_1x \,/\, a_0+a_1x+a_2x^2 \,/\, a_0+a_1x+a_2x^2+a_3x^3$, where $a_j \in \mathbb{R}$.

Def: A subset $U \subset V$ of a vector space V is called a subspace of V if $\alpha u + \beta v \in U \quad \forall u,v \in U, \forall \alpha,\beta \in \mathbb{R}$

Ex: Let $V = \mathbb{R}^3$ and $U = \{(x,y,0) : x,y \in \mathbb{R}\}$. Show that U subspace of V.

(i) $U \subset V$ ok :-)

(ii) Take any $u = (u_1,u_2,0) \in U$ and $v = (v_1,v_2,0) \in U$, $\alpha,\beta \in \mathbb{R}$. One has

$\alpha u + \beta v \underset{\text{Def } u,v}{=} \alpha(u_1,u_2,0) + \beta(v_1,v_2,0) \underset{\text{linear sp.}}{=} (\alpha u_1, \alpha u_2, 0) + (\beta v_1, \beta v_2, 0) =$

$= (\alpha u_1 + \beta v_1, \alpha u_2 + \beta v_2, 0) \in U$:->
$\uparrow$ linear sp. $\quad\quad$ $\uparrow$ def U

$\Rightarrow$ U is a subspace of V.

Ex. Is {polyn. of deg = 1} a subspace of $P^{(2)}(\mathbb{R})$?

No! $\left[\begin{array}{ccc} x & - & x = 0 \\ \text{deg } 1 & 1 & 0 \end{array} \right]$

Ex: Let $V = P^{(5)}(\mathbb{R})$ and $U = P^{(1)}(\mathbb{R})$. Show that $U \subset V$ subspace?

(i) $U \subset V$ OK :->

(ii) Let $p_1(x) = a_0 + a_1 x \in P^{(1)}(\mathbb{R})$, $p_2(x) = b_0 + b_1 x \in P^{(1)}(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$

It is clear that $\alpha p_1(x) + \beta p_2(x) \in U$ (i.e. polyn. deg ≤ 1) OK

$\Rightarrow$ U subspace of V.

Def: Let V be a vector space and $v_1, v_2, ..., v_n \in V$.

A linear combination of these vectors is given by

$a_1 v_1 + a_2 v_2 + .. + a_n v_n$ for some $a_1, a_2, ..., a_n \in \mathbb{R}$.

(S) Ex. Let $V = \mathbb{R}^3$ and $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

All linear comb. of v_1 and v_2 are given by

$a_1 v_1 + a_2 v_2 = \begin{pmatrix} a_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} a_2 \\ 2a_2 \\ 3a_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ 2a_2 \\ 3a_2 \end{pmatrix}$ for all $a_1, a_2 \in \mathbb{R}$

Def. Let V be a vector space and $v_1, v_2, ..., v_n \in V$.

The space of all linear combinations of these vectors is

denoted $\text{span}(v_1, ..., v_n) = \{ a_1 v_1 + a_2 v_2 + .. + a_n v_n : a_1, a_2, ..., a_n \in \mathbb{R} \}$

(S) Ex. $V = \mathbb{R}^2$, $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in V$, $v_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \in V$, $\text{span}(v_1, v_2) = ?$

$\text{span}(v_1, v_2) \underset{\uparrow \text{Def}}{=} \{ a_1 v_1 + a_2 v_2 : a_1, a_2 \in \mathbb{R} \} = \left\{ \begin{pmatrix} a_1 \\ 2a_2 \end{pmatrix} : a_1, a_2 \in \mathbb{R} \right\}$

(S) Ex. Consider the monomials $1, x, x^2$. $\text{span}(1, x, x^2) = ?$

$\text{span}(1, x, x^2) \underset{\uparrow \text{Def span}}{=} \{ a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2 : a_0, a_1, a_2 \in \mathbb{R} \} \underset{\uparrow \text{Def } P^{(2)}(\mathbb{R})}{=} P^{(2)}(\mathbb{R})$

Let us for expl. write the polynomial $4x - 2 \in P^{(2)}(\mathbb{R})$ with

$1, x, x^2$: $4x - 2 = \underbrace{(-2)}_{a_0} \cdot 1 + \underbrace{4}_{a_1} \cdot x + \underbrace{0}_{a_2} \cdot x^2$