We will also consider the following spaces

Def. • Let $1 \leq p < \infty$, one defines

$$L^p(a,b) = \{ f : [a,b] \to \mathbb{R} \; ; \; \|f\|_{L^p} < \infty \}, \text{ where}$$

the L^p-norm is given by

$$\|f\|_{L^p} = \left(\int_a^b |f(x)|^p \, dx \right)^{1/p}$$

• For "$p = \infty$", one has

$$L^\infty(a,b) = \{ f : [a,b] \to \mathbb{R} \text{ (continuous) (bnd)} : \|f\|_\infty < \infty \}$$

with $\|f\|_\infty = \max_{x \in [a,b]} |f(x)|$ (max ess. sup)

essential supremum $|f(x)| \leq L$ a.s. in x

(5) **Ex:** Compute the L^1-norm of $f(x) = 2x + \sin(x)$ defined on $[0, \frac{\pi}{2}]$:

$$\|f\|_{L^1} \underset{\text{Def II.4}}{=} \int_0^{\pi/2} |f(x)| \, dx = \int_0^{\pi/2} |\underbrace{2x}_{\geq 0} + \underbrace{\sin(x)}_{\geq 0}| \, dx = \int_0^{\pi/2} (2x + \sin(x)) \, dx = x^2 - \cos(x) \Big|_0^{\pi/2} =$$

$$= \left(\frac{\pi^2}{4} - 0 \right) - (0 - 1) = \frac{\pi^2}{4} + 1$$

Chapter III: Polynomial approximation in 1D

Goal: We want to approximate functions by polynomials
difficult / easy

Applications: Approx. to sol. to DE. (see later)

1) Some more spaces:

Recall: Space of polyn. on $[a,b]$ of degree $\leq q$:

$$\mathcal{P}^{(q)}(a,b) = \text{span}(1, x, x^2, \dots, x^q)$$

(5) Something else than polyn?

Def. The space of trigonometric polynomials on $[0, L]$ is

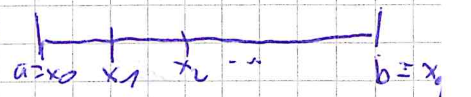
$$T^N(0,L) = \text{span}\left(1, \cos\left(\frac{2\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \dots, \cos\left(\frac{2\pi N}{L} x\right), \sin\left(\frac{2\pi N}{L} x\right)\right) =$$

$$= \left\{ f(x) = \sum_{n=0}^{N} \left(a_n \cos\left(\frac{2\pi}{L} n x\right) + b_n \sin\left(\frac{2\pi}{L} n x\right) \right) ; \; a_n, b_n \in \mathbb{R} \right\}$$

This will be used later.

We now want to write $\mathcal{P}^{(q)}(a,b)$ in ~~another~~ another way.

Consider a grid of $(q+1)$ distinct points



(P) Define Lagrange polynomials by

$$\lambda_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{q} \frac{x-x_j}{x_i-x_j} \quad \text{for } i=0,1,\dots,q$$

⚠ Obs: $\lambda_i(x_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else} \end{cases}$

The space $\mathbb{P}^{(q)}(a,b)$ can also be generated by Lagrange polyn.:

$$\mathbb{P}^{(q)}(a,b) = \operatorname{span}\left(\lambda_0(x), \lambda_1(x), \dots, \lambda_q(x)\right)$$

↑ no proof

We now define another space of interest.

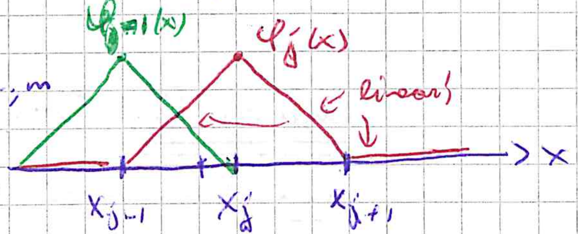
Consider a partition of $[0,1]$ into $m+1$ subintervals

(x_{j-1}, x_j) ; $\mathcal{T}_h : 0 = x_0 < x_1 < x_2 < \dots < x_m < x_{m+1} = 1$, where

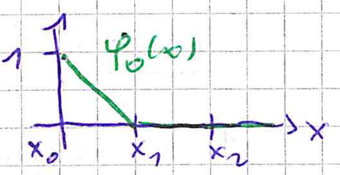
$h_j := x_j - x_{j-1}$.

Consider hat functions $\{\varphi_j\}_{j=0}^{m+1}$ defined by

$$\varphi_j(x) = \begin{cases} \dfrac{x-x_{j-1}}{h_j} & x_{j-1} \leq x \leq x_j \\ \dfrac{x-x_{j+1}}{-h_{j+1}} & x_j \leq x \leq x_{j+1} \\ 0 & \text{else} \end{cases} \qquad j=1,\dots,m$$



In short "$\varphi_j(x_i) = \begin{cases} 1 & i=j \\ 0 & 0 \end{cases}$"



Rem: φ_0 and φ_{m+1} are half hat fct

Def. We define the space of continuous piecewise linear fct on $[0,1]$ by $V_h := V_h(0,1) := \operatorname{span}(\varphi_0, \varphi_1, \dots, \varphi_{m+1})$

⚠ Observe that every $v \in V_h$ can thus be written

$$v(x) = \sum_{j=0}^{m+1} \xi_j \varphi_j(x) \quad \text{with coordinates } \xi_j = v(x_j).$$

2) L^2-projection:

Let $a<b$.

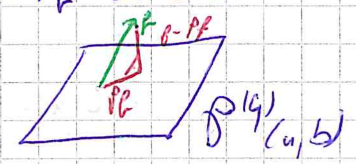
Def. For a given fct $f: [a,b] \to \mathbb{R}$, with $f \in L^2(a,b)$, we define its L^2-projection $Pf \in \mathbb{P}^{(q)}(a,b)$ by

$$\int_a^b f(x)\,p(x)\,dx = \int_a^b (Pf)(x)\,p(x)\,dx \qquad \forall\, p \in \mathbb{P}^{(q)}(a,b).$$

Short: $\quad (f,p)_{L^2} = (Pf,p)_{L^2} \quad \forall\, p \in \mathbb{P}^{(q)}(a,b) \qquad (*)$

or $\quad (f-Pf,p)_{L^2} = 0 \quad \forall\, p \in \mathbb{P}^{(q)}(a,b) \qquad$ remember (orthogonality)

Rem. The last eq. says that the error $f-Pf$ in the L^2-projection is orthogonal to $\mathbb{P}^{(q)}(a,b)$



Rem. Since $\mathbb{P}^{(q)}(a,b) = \mathrm{span}(1,x,x^2,\dots,x^q)$ it is enough to consider $(*)$ for $p(x)=x^j$, $j=0,1,\dots,q$: $\quad (f,x^j) = (Pf,x^j)$ for $j=0,1,\dots,q$.

Ex: Let $f:[a,b]\to\mathbb{R}$ continuous. Find its L^2-projection $Pf(x)\in\mathbb{P}^{(0)}(a,b)$:

We know that $\mathbb{P}^{(0)}(a,b) = \mathrm{span}(1) = \{\text{polyn. degree} \le 0\} = \{a_0 : a_0\in\mathbb{R}\}$

We check $(f,1) = (Pf,1)$ from $(*)$. This is

$$\int_a^b f(x)\,dx = \int_a^b (Pf)(x)\,dx \underset{\uparrow}{=} Pf(x)\cdot(b-a). \text{ Hence } Pf(x) = \frac{1}{b-a}\int_a^b f(x)\,dx \ (\dots)$$

Pf is a constant polyn.

This L^2-proj. gives a kind of average approximation of f.

Prop. (i) The L^2-projection Pf is unique

(ii) Pf is the best approx. of f in $\mathbb{P}^{(q)}(a,b)$ in the L^2-norm

$$\|f-Pf\|_{L^2(a,b)} \le \|f-p\|_{L^2(a,b)} \qquad \forall\, p\in\mathbb{P}^{(q)}(a,b).$$

Proof. (i) Assume that Pf is not unique: $\exists$ 2 polyn. P_1f and P_2f that are L^2-proj. of f: $\quad (f,p) = (P_1f,p)$, $\quad (f,p) = (P_2f,p) \quad \forall\, p\in\mathbb{P}^{(q)}(a,b)$

Hence $(P_1f - P_2f, p) = (f,p) - (f,p) = 0 \quad \forall\, p\in\mathbb{P}^{(q)}(a,b)$.

Take now $p = (P_1f - P_2f)$, one thus gets

$(P_1f-P_2f,\ P_1f-P_2f) = 0$ or $\|P_1f-P_2f\|_{L^2} = 0$ or

$P_1f = P_2f$!! $\quad$ [since $(P_1f-P_2f)^2$ is continuous]

(ii) Let $p \in \mathbb{P}^{(q)}(a,b)$ and consider

$$\|f - Pf\|_{L^2}^2 \underset{\text{Def. norm}}{=} (f - Pf, f - Pf) = (f - Pf, f - p + p - Pf) \underset{\text{linearity}}{=} (f - Pf, f - p) +$$

$$+ (f - Pf, \underbrace{p - Pf}_{\in \mathbb{P}^{(q)}(a,b)}) \underset{\text{orthogonality}}{=} (f - Pf, f - p) + 0 \underset{\text{C-S}}{\leq} \|f - Pf\|_{L^2} \cdot \|f - p\|_{L^2} \underset{f - Pf \neq 0}{\Rightarrow}$$

$$\|f - Pf\|_{L^2} \leq \|f - p\|_{L^2} \quad \forall p \in \mathbb{P}^{(q)}(a,b) \quad (-)$$

3) Galerkin method for IVP:

Read book Chapter 3.2

4) Galerkin FEM for BVP:

We want to find a numerical approximation to solutions to

$$(\text{BVP}) \begin{cases} -u''(x) = f(x) & \text{for } 0 < x < 1 & (\text{DE}) \\ u(0) = 0, \; u(1) = 0, & & (\text{BC}) \end{cases}$$

where f is given and u is unknown.

Rem: This type of BC is called homogeneous Dirichlet BC
$(=0)$

Idea: 1) Multiply DE by test fct v

2) Integrate over domain $(0,1)$ and get a variational formulation

3) Do a Finite Element Method (FEM): to get an approx. $u_h \approx u$.

4) Solve a linear system to find u_h.

Details:

1)+2): Consider the space of test fct $V^0 = \{ v: [0,1] \to \mathbb{R}, \; v, v' \in L^2(0,1)$ and $v(0) = v(1) = 0 \}$

to match the BC.

Multiply DE by $v \in V^0$ and integrate to get

$$-\int_0^1 u''(x)\, v(x)\, dx = \int_0^1 f(x)\, v(x)\, dx$$