Chapter 2: Mathematical tools (summary)

November 5, 2021

Goal: Introduce some (abstract) spaces and various mathematical tools. This will help us to solve (numerically) ordinary and partial differential equations in the next chapters.

- A set V is called a vector space or linear space (VS) if, for all $u, v, w \in V$ and for all $\alpha, \beta \in \mathbb{R}$ one has

  1. $(u+v)+w = u+(v+w) = u+v+w$ (associativity)
  2. $u+v = v+u$ (commutativity)
  3. There exists an element $0 \in V$ such that $u+0 = 0+u = u$ for all $u \in V$ (zero element)
  4. For all $u \in V$, there exists an element $(-u) \in V$ such that $u+(-u) = 0$ (inverse element)
  5. $\alpha(\beta u) = (\alpha\beta)u = \alpha\beta u$ (compatibility)
  6. There exists $1 \in \mathbb{R}$ such that $1u = u$ for all $u \in V$ (identity element)
  7. $\alpha(u+v) = \alpha u + \beta v$ (distributivity)
  8. $(\alpha+\beta)u = \alpha u + \beta u$ (distributivity).

  (Technical comment: the condition (identity element) is mostly needed if one considers VS on something else than $\mathbb{R}$, see the further readings if you are interested).

  The elements in V are called vectors (but they can be something else, like "normal" vectors, matrices, functions, or sequences) and the elements in $\mathbb{R}$ are called scalars. The above axioms (rules) tell us that we can do anything reasonable with vectors and scalars in a VS.

  Example: The vector space of all polynomials, defined on $\mathbb{R}$, of degree $\leq n$ is denoted by

  $$\mathscr{P}^{(n)}(\mathbb{R}) = \left\{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n : \ a_0, a_1, \ldots, a_n \in \mathbb{R}\right\}.$$

- Let V be a VS. A subset $U \subset V$ is called a subspace of V if $\alpha u + \beta v \in U$ for all $u, v \in U$ and $\alpha, \beta \in \mathbb{R}$.

- Let V be a VS. The space of all linear combinations of the elements $v_1, v_2, \ldots, v_n \in V$ is denoted by

  $$\operatorname{span}(v_1, \ldots, v_n) = \{a_1 v_1 + a_2 v_2 + \ldots + a_n v_n : \ a_1, \ldots, a_n \in \mathbb{R}\}.$$

  Example: $\operatorname{span}\left(1, x, x^2\right) = \left\{a_0 1 + a_1 x + a_2 x^2 : \ a_0, a_1, a_2 \in \mathbb{R}\right\} = \mathscr{P}^{(2)}(\mathbb{R})$.

- A set $\{v_1, v_2, \ldots, v_n\}$ in a VS V is linearly independent if the equation

  $$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0 \in V$$

  has only the trivial solution $a_1 = a_2 = \ldots = a_n = 0 \in \mathbb{R}$. Else it is called linearly dependent.

  Example: The set $\{1, x, x^2\} \subset \mathscr{P}^{(2)}(\mathbb{R})$ is linearly independent.

- A set $\{v_1, v_2, \ldots, v_n\}$ in a VS V is called a basis of V if the set is linearly independent and $\operatorname{span}(v_1, \ldots, v_n) = V$. The dimension of V is then given by the number of elements of this set, here $\dim(V) = n$.

  Example: The set $\{1, x, x^2\}$ is a basis of $\mathscr{P}^{(2)}(\mathbb{R})$ and thus $\dim(\mathscr{P}^{(2)}(\mathbb{R})) = 3$.

- A scalar product or inner product on a VS V is a map $(\cdot,\cdot)\colon V \times V \to \mathbb{R}$ such that, for all $u, v, w \in V$ and $\alpha \in \mathbb{R}$, one has

1. $(u,v) = (v,u)$ (symmetry)
2. $(u+\alpha v, w) = (u,w) + \alpha(v,w)$ (linearity)
3. $(u,u) \geq 0$ (positivity)
4. $(u,u) = 0 \in \mathbb{R}$ if and only if $u = 0 \in V$.

- A VS V with an inner product is called an inner product space, which is denoted by $(V,(\cdot,\cdot))$ or $(V,(\cdot,\cdot)_V)$ or $(V,\langle\cdot,\cdot\rangle_V)$.

  Such space has a norm defined by $\|v\| = \sqrt{(v,v)}$ for all $v \in V$.

  Example: The space of square integrable functions defined on the interval $[a,b]$ is denoted by

  $$L^2([a,b]) = L^2(a,b) = L_2(a,b) = \left\{ f\colon [a,b] \to \mathbb{R} \colon \int_a^b |f(x)|^2 \,\mathrm{d}x < \infty \right\}.$$

  It is equipped with the inner product

  $$(f,g)_{L^2} = \int_a^b f(x)g(x)\,\mathrm{d}x$$

  which induces the norm

  $$\|f\|_{L^2} = \sqrt{(f,f)_{L^2}} = \sqrt{\int_a^b |f(x)|^2 \,\mathrm{d}x}.$$

- Let $(V,(\cdot,\cdot))$ be an inner product space and $u,v \in V$. The elements u and v are called orthogonal if $(u,v) = 0$. Notation: $u \perp v$.

- Let $(V,(\cdot,\cdot))$ be an inner product space and $u,v \in V$. Cauchy–Schwarz inequality (CS) reads

  $$|(u,v)| \leq \|u\| \cdot \|v\|.$$

- Let $(V,(\cdot,\cdot))$ be an inner product space and $u,v \in V$. The triangle inequality ($\triangle$) reads

  $$\|u+v\| \leq \|u\| + \|v\|.$$

- The space of continuous function defined on $[a,b]$ is given by

  $$C^0([a,b]) = \mathscr{C}^0([a,b]) = \mathscr{C}^{(0)}(a,b) = \{f\colon [a,b] \to \mathbb{R} : f \text{ is continuous}\}$$

  and equipped with the norm

  $$\|f\|_{C^0([a,b])} = \max_{a \leq x \leq b} |f(x)|.$$

  (Technical observation: one should have a suppremum sup *in place of* max*, see further resources if you are interested. We will try to avoid this technicality in the lecture, hopefully.)*

  Similarly, one can define the space of continuously differentiable functions

  $$C^1([a,b]) = \mathscr{C}^1([a,b]) = \mathscr{C}^{(1)}(a,b) = \{f\colon [a,b] \to \mathbb{R} : f, f' \text{ are continuous}\}$$

  and equipped with the norm

  $$\|f\|_{C^1([a,b])} = \|f\|_{C^0([a,b])} + \|f'\|_{C^0([a,b])} = \max_{a \leq x \leq b} (|f(x)| + |f'(x)|).$$

  Similarly, one can also define the space $C^k([a,b])$ of k time continuously differentiable functions.

  (Technical observation: The above definition should be enough for us, but observe that a precise definition of the above space can be found in the further resources below.)

- For $1 \leq p < \infty$, we consider the spaces

$$L^p([a,b]) = L_p(a,b) = \left\{ f\colon [a,b] \to \mathbb{R} : \|f\|_{L^p} < \infty \right\},$$

  with the L^p-norm

$$\|f\|_{L^p} = \left(\int_a^b |f(x)|^p \, \mathrm{d}x \right)^{1/p}.$$

  For "$p = \infty$", one has

$$L^\infty([a,b]) = L_\infty(a,b) = \left\{ f\colon [a,b] \to \mathbb{R} : \|f\|_{L^\infty} < \infty \right\},$$

  with the L^∞-norm

$$\|f\|_{L^\infty} = \max_{a \leq x \leq b} |f(x)|.$$

  (Technical observation: one should have an ess.sup in place of the maximum. But you can forget this comment for the present lecture. We will stay in the easiest possible situations.)

Further resources:

- sv.wikipedia.org/LinjärtRum
- web.auburn.edu/InnerProduct
- sv.wikipedia.org/InreProduktrum
- sv.wikipedia.org/Lp
- sv.wikipedia.org/CS
- math.carleton.ca/Basis
- brilliant.org/Basis
- terrytao/SpaceOfFunctions
- ljll.math.upmc.fr/SpaceOfFunctions