Integrate by part:

$$-u'(x)v(x)\Big|_0^1 + \int_0^1 u'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx \iff$$

(S) $$-u'(1)\underbrace{v(1)}_{=0} + u'(0)\underbrace{v(0)}_{=0} + \int_0^1 u'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx$$

by def of V^0

This gives the variational formulation (VF):

(VF) Find $u \in V^0$ s.t. $\int_0^1 u'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx \quad \forall v \in V^0$

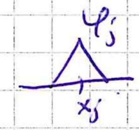
Hat, 1-2-3
BVP, VF, FE

3) To get a FEM, we specify a finite dimensional test space $V_h^0 \subset V^0$:

Consider uniform partition of $[0,1]$: $\mathcal{T}_h: 0 = x_0 < x_1 < \dots < x_{m+1} = 1$, where $h = x_j - x_{j-1}$ for $j = 1, 2, \dots, m+1$.

Consider FE space $V_h^0 = \{v: [0,1] \to \mathbb{R} : v$ cont. pw linear on $\mathcal{T}_h$, $v(0) = v(1) = 0\}$.

Observe that $V_h^0 = \text{span}(\varphi_1, \varphi_2, \dots, \varphi_m)$ with hat fct



The FE problem then reads:

(FE) Find $u_h \in V_h^0$ s.t. $\int_0^1 u_h'(x)\chi'(x)\,dx = \int_0^1 f(x)\chi(x)\,dx \quad \forall \chi \in V_h^0$

To find a linear syst. of eq. to compute u_h, one choose

(S) $\chi = \varphi_i$ for $i = 1, 2, \dots, m$ and insert $u_h(x) = \sum_{j=1}^m \xi_j \varphi_j(x)$ into (FE):

φ_j basis   unknown

4) This gives us $\int_0^1 \left(\sum_{j=1}^m \xi_j \varphi_j'(x)\right)\varphi_i'(x)\,dx = \int_0^1 f(x)\varphi_i(x)\,dx$ for $i = 1, \dots, m$

or $$\sum_{j=1}^m \xi_j \left(\int_0^1 \varphi_j'(x)\varphi_i'(x)\,dx\right) = \int_0^1 f(x)\varphi_i(x)\,dx$$

or linear syst. $A\xi = b$, where

A is called the stiffness matrix with entries $A_{ij} = \int_0^1 \varphi_j'(x)\varphi_i'(x)\,dx$

b is the load vector with entries $b_i = \int_0^1 f(x)\varphi_i(x)\,dx$

and $\xi = \begin{pmatrix}\xi_1\\ \vdots\end{pmatrix}$ unknown.

Remembering the def. of the hat fct

$$\varphi_j(x) = \begin{cases} \frac{x - x_{j-1}}{h} & x_{j-1} \le x \le x_j \\ \frac{x - x_{j+1}}{-h} & x_j \le x \le x_{j+1} \\ 0 & \text{else} \end{cases} \qquad \text{for } j = 1, 2, \ldots, m$$

(5) one gets $\varphi_j'(x) = \begin{cases} 1/h & x_{j-1} \le x \le x_j \\ -1/h & x_j \le x < x_{j+1} \\ 0 & \text{else} \end{cases}$

and for instance

the entry A_{jj} reads $A_{jj} \underset{\text{def}}{=} \int_0^1 (\varphi_j'(x))^2 dx \underset{\text{partition}}{=} \int_0^{x_1} \underbrace{(\varphi_j'(x))^2}_{0} dx +$

$$+ \int_{x_1}^{x_2} \underbrace{(\varphi_j'(x))dx}_{0} + \ldots + \int_{x_{j-1}}^{x_j} \underbrace{(\varphi_j'(x))^2}_{1/h^2} dx + \int_{x_j}^{x_{j+1}} \underbrace{(\varphi_j'(x))^2}_{1/h^2} dx + \ldots + \int_{x_m}^{1} \underbrace{(\varphi_j(x))^2}_{0} dx =$$

$$= \frac{1}{h^2} \cdot h + \frac{1}{h^2} \cdot h = \frac{2}{h} //$$

...

One then obtains the stiffnessmatrix

$$A = \frac{1}{h} \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}$$

(verify the rest @ home)

For the load vector b, we have to compute $b_j = \int_0^1 f(x)\varphi_j(x)dx$. This may be difficult in general

⟶ need a numerical approximation of integrals (→ next chapter)

Once this is done, one solves the linear system $A\xi = b$ to get ξ and then $u_h(x) = \sum_{j=1}^m \xi_j \varphi_j(x)$ an approx. of the sol. to our BVP!!

Chapter IV: Interpolation and numerical integration in 1D

Goal: Interpol: Pass a simple fct through a given set of data points

Num. integ: Use the above to find approx. of $\int_a^b f(x)dx$

Better condition than classical basis

Ex: Linear interpolation on $[0,1]$ (Lagrange basis)

One has $x_0 = 0$, $x_1 = 1$ and $q = 1$.

We know $\lambda_0(x) = \frac{x - x_1}{x_0 - x_1} = 1 - x$ and $\lambda_1(x) = \frac{x - x_0}{x_1 - x_0} = x$

for Lagrange's polyn.

In addition, $\mathbb{P}^{(1)}[0,1] = \text{span}(\lambda_0, \lambda_1) = \text{span}(1-x, x)$

Hence, we can write $\Pi_1 f(x) = b_0 \cdot \overbrace{(1-x) + b_1}^{\text{basis}} \cdot x$

coeff?

We get the system:

$\Pi_1 f(x_0) \overset{!}{=} f(x_0) \Leftrightarrow \Pi_1 f(0) = f(0) \Leftrightarrow b_0 = f(0)$

$\Pi_1 f(x_1) \overset{!}{=} f(x_1) \Leftrightarrow \Pi_1 f(1) = f(1) \Leftrightarrow b_1 = f(1)$

$\hookrightarrow \Pi_1 f(x) = f(0) \cdot (1-x) + f(1)\, x$ //

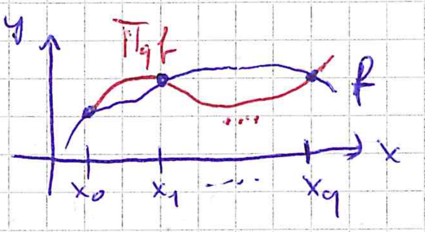
Rem: Same result as before :-)

1) Polynomial interpolation:

Idea / Def:

Consider a cont. fct $f: [a,b] \to \mathbb{R}$ and $(q+1)$ distinct interpolation points $(x_i, f(x_i))_{i=0}^{q}$ with $a = x_0 < x_1 < \dots < x_q = b$.

A polyn. $\Pi_q f \in \mathbb{P}^{(q)}(a,b)$ is an interpolant for f if $\Pi_q f(x_j) = f(x_j)$ for $j = 0, 1, 2, \dots, q$.



Ex: Linear interpolation on $[0,1]$ (classical basis)

One has 2 points $x_0 = 0$ and $x_1 = 1$, hence $q = 1$.

Since $\mathbb{P}^{(q)}(0,1) = \text{span}(1, x)$, one can write

$$\Pi_1 f(x) = \underbrace{a_0 \cdot \overbrace{1 + a_1 \cdot x}^{\text{basis}}}_{\text{unknown coeff.}}$$

5 We have a system w/ 2 eqs for 2 unknowns:

$\Pi_1 f(x_0) \overset{!}{=} f(x_0) \Leftrightarrow \Pi_1 f(0) = f(0) \Leftrightarrow a_0 = f(0)$

$\Pi_1 f(x_1) \overset{!}{=} f(x_1) \Leftrightarrow \Pi_1 f(1) = f(1) \Leftrightarrow a_0 + a_1 = f(1) \Rightarrow a_1 = f(1) - f(0)$

$\Rightarrow \Pi_1 f(x) = f(0) + (f(1) - f(0))\, x$ // (eq. of a line...)

Add expl. p. 7'

Same for $q = 2, 3$, etc cf. exo.

What is the error of such interpolation polynomials?

Recall:

L^p, L^∞ norms: $\|f\|_{L^p(a,b)} = \left(\int_a^b |f(x)|^p \, dx \right)^{1/p}$ and $\|f\|_{L^\infty(a,b)} = \max_{a \le x \le b} |f(x)|$

Hence, the "distance" between 2 fct is $\|f-g\|_{L^p}$ or $\|f-g\|_{L^\infty}$.

Th: Let $p = 1, 2, \infty$. Assume $f'' \in L^p(a,b)$. Then, $\exists$ constants c_1, c_2, c_3 such that

1) $\| \Pi_1 f - f \|_{L^p(a,b)} \leq C_1 \cdot (b-a)^2 \cdot \| f'' \|_{L^p(a,b)}$

2) $\| \Pi_1 f - f \|_{L^p(a,b)} \leq C_2 \cdot (b-a) \, \| f'' \|_{L^p(a,b)}$

3) $\| (\Pi_1 f)' - f' \|_{L^p(a,b)} \leq C_3 \cdot (b-a) \cdot \| f'' \|_{L^p(a,b)}$

Proof: (of 1) for $p = \infty$)

From previous expl, we know that

$\Pi_1 f(x) = f(\overset{a}{x_0}) \lambda_0(x) + f(\overset{b}{x_1}) \lambda_1(x)$

Next, we use a Taylor expansion of $f(x_0)$ and $f(x_1)$ about x:

$f(x_0) = f(x) + (x_0 - x) f'(x) + \frac{(x_0 - x)^2}{2!} f''(\xi_0)$, for some $\xi_0 \in (a, x)$

$f(x_1) = f(x) + (x_1 - x) f'(x) + \frac{(x_1 - x)^2}{2!} f''(\xi_1)$, " " $\xi_1 \in (x, b)$.

Insert into $\Pi_1 f(x)$ gives

$\Pi_1 f(x) = \dots = f(x) + \frac{1}{2}(x_0 - x)^2 f''(\xi_0) \lambda_0(x) + \frac{1}{2}(x_1 - x)^2 f''(\xi_1) \lambda_1(x)$.

Finally, for the error, one gets

$$|\Pi_1 f(x) - f(x)| \overset{\Delta}{\leq} \frac{1}{2} \underbrace{|x - x_0|^2}_{\substack{\leq |b-a|^2 \\ \text{since } a \leq x \leq b}} \cdot \underbrace{|f''(\xi_0)|}_{\leq \max\limits_{a \leq x \leq b} |f''(x)|} \cdot \overbrace{|\lambda_0(x)|}^{\leq 1} + \frac{1}{2} \underbrace{|x_1 - x|^2}_{\leq |b-a|^2} \underbrace{|f''(\xi_1)|}_{\leq \max\limits_{a \leq x \leq b} |f''(x)|} \cdot \overbrace{|\lambda_1(x)|}^{\leq 1}$$

and $\| \Pi_1 f - f \|_{L^\infty(a,b)} \leq (b-a)^2 \cdot \max\limits_{a \leq x \leq b} |f''(x)| \leq (b-a)^2 \| f'' \|_{L^\infty(a,b)}$:-) $\boxtimes$

2) Continuous piecewise linear interpolation:

Here, we consider pw interpolant instead of a polyn. as in previous section

Recall: $V_h = \{ v : [a,b] \to \mathbb{R} : v \text{ is cont. pw linear on } \mathcal{T}_h \} = \operatorname{span}(\varphi_0, \varphi_1, \dots, \varphi_{m+1})$

where $\mathcal{T}_h$ is a uniform partition of $[a,b]$ with $x_j - x_{j-1} = h$

and φ_j are hat fcts.

For $v \in V_h$, one thus has $v(x) = \sum\limits_{j=0}^{m+1} \xi_j \varphi_j(x)$

Def: Let $f : [a,b] \to \mathbb{R}$ continuous and the above partition $\mathcal{T}_h$.

The continuous piecewise linear interpolant of f is defined by