

Chapter 4: Interpolation and numerical integration in 1d (summary)

November 15, 2021

Goals:

Interpolation: We want to pass a (simple) function through a given set of data points.

Numerical integration: We want to find numerical approximations of integrals $\int_a^b f(x) dx$.

- Let $q \in \mathbb{N}$. Consider a continuous function $f: [a, b] \rightarrow \mathbb{R}$ and $q + 1$ distinct interpolation points $(x_j, f(x_j))_{j=0}^q$ with $a = x_0 < x_1 < \dots < x_q = b$. A polynomial $\pi_q f \in \mathcal{P}^{(q)}(a, b)$ is an **interpolant for f** if

$$\pi_q f(x_j) = f(x_j) \quad \text{for } j = 0, 1, 2, \dots, q.$$

Examples (2 ways of seeing the interpolant π_q): Remembering that $\mathcal{P}^{(q)}(a, b) = \text{span}(1, x, x^2, \dots, x^q)$, one gets the polynomial interpolant

$$\pi_q f(x) = \sum_{j=0}^q a_j x^j.$$

Taking $\mathcal{P}^{(q)}(a, b) = \text{span}(\lambda_0(x), \lambda_1(x), \dots, \lambda_q(x))$, one gets the **Lagrange interpolant**

$$\pi_q f(x) = \sum_{j=0}^q f(x_j) \lambda_j(x),$$

where λ_j are the Lagrange polynomials seen in the previous chapter.

- Consider a uniform partition of an interval $[a, b]$, denoted $\tau_h: x_0 = a < x_1 < \dots < x_{m+1} = b$, and the space of continuous piecewise linear functions on τ_h , $V_h = \text{span}(\varphi_0, \dots, \varphi_{m+1})$ with hat functions φ_j . The **continuous piecewise linear interpolant of f** is defined by

$$\pi_h f(x) = \sum_{j=0}^{m+1} f(x_j) \varphi_j(x) \quad \text{for } x \in [a, b].$$

If $f \in \mathcal{C}^2(a, b)$ one has, for instance, the following bound for the interpolation error for the continuous piecewise linear interpolant

$$\|\pi_h f - f\|_{L^p(a, b)} \leq Ch^2 \|f''\|_{L^p(a, b)},$$

for $p = 1, 2, \infty$.

- Let us give 3 classical **quadrature rules** to numerically approximate the integral $\int_a^b f(x) dx$:

The **midpoint rule** reads

$$\int_a^b f(x) dx \approx (b - a) f\left(\frac{a + b}{2}\right).$$

The **trapezoidal rule** reads

$$\int_a^b f(x) dx \approx \frac{b - a}{2} (f(a) + f(b)).$$

The **Simpson rule** reads

$$\int_a^b f(x) \, dx \approx \frac{b-a}{6} \left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right).$$

In practice, one first considers a (uniform) partition of the interval $[a, b]$, $a = x_0 < x_1 < \dots < x_N = b$, and then apply a quadrature rule (denoted by $QF(x_j, x_{j+1}, f)$ below) on each small subintervals:

$$\int_a^b f(x) \, dx = \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} f(x) \, dx \approx \sum_{j=0}^{N-1} QF(x_j, x_{j+1}, f).$$

Further resources:

- www.dcode.fr
- www.maths.lth.se1, www.maths.lth.se2, www.maths.lth.se3.
- www.phys.libretexts.org
- www.khanacademy.org
- tutorial.math.lamar.edu