The backward difference formulas provides

Backward Euler scheme: $y_{\ell+1} = y_\ell + k f(y_{\ell+1})$

Similar to above, one could consider

(P) Crank-Nicolson scheme: $y_{\ell+1} = y_\ell + \frac{k}{2}\left(f(y_\ell) + f(y_{\ell+1})\right)$

average of f

Ex: Consider

IVP $\begin{cases} \dot{y}(t) = 5y(t) & , 0 < t < T \\ y(0) = 1 \end{cases}$

Define stepsize $k = \frac{T}{N}$ for some large N. (ex. $N = 100$)

(i) One step of forward Euler reads

(S) $$y_1 \underset{\text{Def Euler}}{=} y_0 + k \cdot f(y_0) \underset{\text{Def IVP}}{=} y_0 + k \cdot 5 \cdot y_0 \underset{\text{Def IC}}{=} 1 + k \cdot 5 \cdot 1 = 1 + 5k \approx y(k)$$

(ii) One step of backward Euler reads

$$y_1 = y_0 + k \cdot f(y_1) = 1 + 5k y_1 \quad \text{or} \quad y_1 = \frac{1}{1-5k} \approx y(k)$$

(iii) One step of Crank-Nicolson reads:

$$y_1 = y_0 + \frac{k}{2}\left(f(y_0) + f(y_1)\right) = 1 + \frac{k}{2} \cdot 5(y_0 + y_1) \quad \text{or}$$

$$y_1 = \frac{1 + 5k/2 \, y_0}{1 - 5k/2}$$

Ex: (Application to PDE, see also lab)

Consider the following system of linear IVP

$$\begin{cases} M \cdot \dot{\xi}(t) + S \xi(t) = F(t) & , 0 \le t < T \\ \xi(0) = \xi_0 \end{cases}$$

where M is an $m \times m$ matrix (ex. mass matrix)

S " " " (ex. stiffness matrix)

$F(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_m(t) \end{pmatrix}$ is a vector (ex. load vector)

$\xi(t) = \begin{pmatrix} \xi_1(t) \\ \vdots \\ \xi_m(t) \end{pmatrix}$ is the unknown vector

(i) Euler scheme applied to above IVP gives

$$M\left(\frac{\xi^{(\ell+1)}-\xi^{(\ell)}}{k}\right)+S'\xi^{(\ell)}=F(t_\ell) \text{ or } M\xi^{(\ell+1)}=(M-kS')\xi^{(\ell)}+kF(t_\ell)$$

or $\quad \xi^{(0)}=\xi(0)$

$t_{\ell+1}=t_\ell+k \qquad$ for $\ell=0,1,2,\ldots$

$M\xi^{(\ell+1)}=(M-kS')\xi^{(\ell)}+kF(t_\ell) \qquad$ where $\xi^{(\ell)}\approx\xi(t_\ell)$

(ii) Backward Euler reads

$$M\left(\frac{\xi^{(\ell+1)}-\xi^{(\ell)}}{k}\right)+S'\xi^{(\ell+1)}=F(t_{\ell+1}) \text{ or } (M+kS')\xi^{(\ell+1)}=M\xi^{(\ell)}+kF(t_{\ell+1})$$

That is the recursion

$\xi^{(0)}=\xi(0)$

$(M+kS')\xi^{(\ell+1)}=M\xi^{(\ell)}+kF(t_{\ell+1})$

(iii) CN reads

$\xi^{(0)}=\xi(0)$

$$\left(M+\frac{k}{2}S'\right)\xi^{(\ell+1)}=\left(M-\frac{k}{2}S'\right)\xi^{(\ell)}+\frac{k}{2}\left(F(t_{\ell+1})+F(t_\ell)\right)$$

Chapter VII: PDE in 1D

Goal: Use FEM and above time integrator to numerically discretise heat and wave equation in 1D ($u(x,t)$)

FEM $\quad$ Euler/CN

1) Heat equation in 1D:

Model problem to describe heat flow along a thin wire/metal rod of length 1:

$$(H)\begin{cases} u_t(x,t)-u_{xx}(x,t)=f(x,t) & 0<x<1,\ t>0 & \text{DE} \\ u(0,t)=u_x(1,t)=0 & t>0 & \text{BC} \\ u(x,0)=u_0(x) & 0<x<1, & \text{IC} \end{cases}$$

where $u(x,t)\to$ temperature at point x, time t (unknown)

$u_0(x) \to$ initial temperature profile

$u(0,t)=0 \to$ fixed temperature 0 at $x=0$ (hom. Dirichlet BC)

$u_x(1,t)=0 \to$ insulated boundary at $x=1$ (no heat flux)
(hom. Neumann BC)

$f(x,t) \longrightarrow$ heat sources / sinks

Office!

Th. The sol. to the heat eq. (H) satisfies the stability estimates

(i) $\|u(\cdot,t)\|_{L^2(0,1)} \le \|u_0\|_{L^2(0,1)} + \int_0^t \|f(\cdot,s)\|_{L^2(0,1)}\,ds$

(ii) $\|u_x(\cdot,t)\|_{L^2(0,1)} \le \|u_0'\|_{L^2(0,1)} + \int_0^t \|f(\cdot,s)\|_{L^2(0,1)}\,ds$

(iii) When $f\equiv 0$, one has $\|u(\cdot,t)\|_{L^2(0,1)} \le \|u_0\|_{L^2(0,1)}\cdot e^{-t}$

Rem: • Notation $\|u(\underbrace{\cdot}_{\text{integrate wrt } x},t)\|_{L^2(0,1)}$ means integrate w/ respect to x: $\left(\int_0^1 u(x,t)^2dx\right)^{1/2}$

• (i) is used to show uniqueness of sol. to (H)

• (iii) says: "mean temperature" $\searrow$ in time

2) Discretisation of the heat equation:

Consider model problem

$$(H)\quad \begin{cases} u_t(x,t)-u_{xx}(x,t)=f(x,t) & 0<x<1,\ 0<t\le T & (DE)\\ u(0,t)=u(1,t)=0 & 0<t\le T & (BC)\\ u(x,0)=u_0(x) & 0<x<1 & (IC)\end{cases}$$

(S) Idea: FEM in space $\rightsquigarrow$ linear syst. of ODE $\rightsquigarrow$ backward Euler in time.

(i) To get a VF, consider the following test/trial space

$H_0^1=\{v:[0,1]\to\mathbb{R}\ :\ v,v'\in L^2(0,1),\ v(0)=v(1)=0\}$

Multiply DE with $v=v(x)\in H_0^1$, integrate over $(0,1)$, by parts:

$$\int_0^1 u_t(x,t)v(x)dx-\underbrace{\int_0^1 u_{xx}(x,t)v(x)dx}_{\underbrace{u_x(x,t)v(x)\Big|_0^1}_{=0 \text{ since } v\in H_0^1}-\int_0^1 u_x(x,t)v_x(x)dx}=\int_0^1 f(x,t)v(x)dx$$

The VF reads: For all $t \in (0,T]$:

(VF) Find $u(\cdot,t) \in H_0^1$ s.t. $(u_t(\cdot,t), v)_{L^2} + (u_x(\cdot,t), v_x)_{L^2} = (f(\cdot,t), v)_{L^2}$ $\forall v \in H_0^1$

$u(x,0) = u_0(x)$

(ii) Next, discretise in space using FEM:

Consider uniform partition of $[0,1]$, T_h: $x_0 = 0 < x_1 < x_2 < \dots < x_{m+1} = 1$,

with mesh $h = x_j - x_{j-1} = \frac{1}{m+1}$.

Consider space $V_h^0 = \{v : [0,1] \to \mathbb{R} : v$ cont, pw linear on T_h, $v(0) = v(1) = 0\} = \text{span}(\varphi_1, \varphi_2, \dots, \varphi_m)$

with hat fct φ_j.

The FE problem reads: For all $t \in (0,T]$:

(FE) Find $u_h(\cdot,t) \in V_h^0$ s.t. $(u_{h,t}(\cdot,t), \chi)_{L^2} + (u_{h,x}(\cdot,t), \chi_x)_{L^2} = (f(\cdot,t), \chi)_{L^2}$

$u_h(x,0) = \Pi_h u_0(x)$ $\forall \chi \in V_h^0$.

Here, $\Pi_h u_0$ is the cont. pw linear interpol. of u_0.

(iii) We get a linear syst. of ODE by choosing

test fct $\chi = \varphi_i$, $i = 1,2,\dots,m$, and writing

$$u_h(x,t) = \sum_{j=1}^m \underbrace{\xi_j(t)}_{\text{time dependent unknowns}} \varphi_j(x)$$

Insert above into (FE) to get:

$$\begin{cases} \sum_{j=1}^m \dot{\xi}_j(t) \underbrace{(\varphi_i, \varphi_j)_{L^2}}_{m_{ij}} + \sum_{j=1}^m \xi_j(t) \underbrace{(\varphi_i', \varphi_j')_{L^2}}_{s_{ij}} = \underbrace{(f(\cdot,t), \varphi_i)_{L^2}}_{F_i(t)} \quad \text{for } i = 1,2,\dots,m. \\ u_h(x,0) = \Pi_h u_0(x) \Leftrightarrow \sum_{j=1}^m \xi_j(0) \varphi_j(x) \overset{!}{=} \sum_{j=1}^m u_0(x_j) \varphi_j(x) \end{cases}$$

This gives

$$\text{(ODE)} \quad \begin{cases} M \dot{\xi}(t) + S \xi(t) = F(t) \\ \xi(0) = \xi_0, \end{cases}$$

where $M \to$ mass matrix, $S \rightsquigarrow$ stiffness matrix