$$F(t) = \begin{pmatrix} F_1(t) \\ \vdots \\ F_m(t) \end{pmatrix} \quad \text{and} \quad \xi_0 = \begin{pmatrix} u_0(x_1) \\ \vdots \\ u_0(x_m) \end{pmatrix}$$

(iv) To find an approx. of $\xi(t)$, we use a time integrator, for instance backward Euler with step size $k = \frac{T}{N}$.

We get (see above)

$$M\left(\frac{\xi^{(\ell+1)} - \xi^{(\ell)}}{k}\right) + S\,\xi^{(\ell+1)} = F(t_{\ell+1}) \quad , \xi^{(0)} = \xi_0$$

or

$$\begin{cases} (M + kS)\,\xi^{(\ell+1)} = M\xi^{(\ell)} + k\,F(t_{\ell+1}) \text{ for } \ell = 0,1,2,\dots,N-1 \\ \xi^{(0)} = \xi_0 \end{cases}$$

5 Rem. • If the integrals in $F(t)$ are complicated
$\rightsquigarrow$ quadrature formula

• One has to solve a linear syst. of eq. at each time step $(\ell = 0, 1, \dots, N)$

This gives us $\xi^{(\ell)} \approx \xi(t_\ell)$ where $t_\ell = \ell \cdot k$ and then the numerical approx $u_h^k(x, t_\ell) \approx u(x, t_\ell)$ sol. to (H)!!

3) The wave eq. in 1d:

Simple model to describe a vibrating violon string $(0 \leq x \leq 1,\ 0 \leq t \leq T)$.

$$(W)\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = f(x,t) & \text{DE} \\ u(0,t) = u(1,t) = 0 & \text{BC} \\ u(x,0) = u_0(x) & \\ u_t(x,0) = v_0(x) & \end{cases}$$

IC $\to$ position, $\to$ velocity

Th: For a homogeneous problem, i.e. $f=0$ in (W), one has conservation of energy:

$$\underbrace{\frac{1}{2}\|u_t(\cdot,t)\|_{L^2}^2}_{\text{kinetic}} + \underbrace{\frac{1}{2}\|u_x(\cdot,t)\|_{L^2}^2}_{\text{potential}} = \underbrace{\frac{1}{2}\|v_0\|_{L^2}^2 + \frac{1}{2}\|u_0'\|_{L^2}^2}_{\text{initial energy}} \equiv \text{const} \ \forall t$$

Proof:

- Multiply DE with $u_t(x,t)$, integrate over $(0,1)$ and by part:

$$\int_0^1 \underbrace{u_{tt}\cdot u_t}_{\frac{1}{2}\frac{d}{dt}(u_t^2)}\,dx - \underbrace{\int_0^1 u_{xx}\cdot u_t\,dx}_{\underbrace{u_x u_t\Big|_{x=0}^{1}}_{\substack{=0 \\ \text{(Dirichlet BC)}}} - \int_0^1 \underbrace{u_x\cdot u_{tx}}_{\frac{1}{2}\frac{d}{dt}(u_x^2)}\,dx} = 0$$

This gives $\frac{1}{2}\frac{d}{dt}\|u_t\|_{L^2(0,1)}^2 + \frac{1}{2}\frac{d}{dt}\|u_x\|_{L^2(0,1)}^2 = 0$

or $\frac{1}{2}\|u_t\|_{L^2}^2 + \frac{1}{2}\|u_x\|_{L^2}^2 = \text{const.}$ (int. in t)

Rem: Introducing a new variable for the velocity $v = u_t$, one can rewrite (W) as a syst. of first order DE:

$$\begin{cases} u_t = v \\ v_t \underset{\text{Def } v}{=} u_{tt} \underset{\text{Def }(W)}{=} u_{xx} + f \end{cases} \quad \text{or} \quad \begin{pmatrix} u \\ v \end{pmatrix}_t = \underbrace{\begin{pmatrix} 0 & 1 \\ \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} u \\ v \end{pmatrix}}_{w} + \underbrace{\begin{pmatrix} 0 \\ f \end{pmatrix}}_{F}$$

or $w_t = Aw + F$ (see later)

4) Discretisation of the wave eq. in 1d:

Idea: Apply FEM in space and CN in time (or backward Euler) to get a numerical approx. to u, sol. to wave eq. (W).

(i) To find a VF, consider $H_0^1 = \{v : [0,1] \to \mathbb{R} : v, v' \in L^2(0,1),\ v(0) = v(1) = 0\}$ (20)

and multiply DE by test fct, integrate over (0,1) and by parts:

$$\int_0^1 u_{tt}(x,t)v(x)\,dx \underbrace{- \int_0^1 u_{xx}(x,t)v(x)\,dx}_{\int_0^1 u_x(x,t)v_x(x)\,dx} = \int_0^1 f(x,t)v(x)\,dx \quad \forall v \in H_0^1$$

This gives:

(VF) Find $u \in H_0^1$ s.t., for every $t \in (0,T]$,

$$\begin{cases} (u_{tt}(\cdot,t), v)_{L^2} + (u_x(\cdot,t), v_x)_{L^2} = (f(\cdot,t), v)_{L^2} & \forall v \in H_0^1 \\ u(x,0) = u_0(x) \\ u_t(x,0) = v_0(x) \end{cases}$$

(ii) For the FE problem, consider $V_h^0 = \text{span}(\varphi_1, \dots, \varphi_m)$ and get

(FE) Find $u_h(\cdot,t) \in V_h^0$ s.t., for every $t \in (0,T]$,

$$\begin{cases} (u_{h,tt}(\cdot,t), \chi)_{L^2} + (u_{h,x}(\cdot,t), \chi_x)_{L^2} = (f(\cdot,t), \chi)_{L^2} \ \forall \chi \in V_h^0 \\ u_h(x,0) = \Pi_h u_0(x) \\ u_{h,t}(x,0) = \Pi_h v_0 \end{cases}$$

(iii) To get a linear syst. of ODEs, choose $\chi = \varphi_i$, $i = 1, \dots, m$,

and write $u_h(x,t) = \sum_{j=1}^m \zeta_j(t)\varphi_j(x)$ into (FE). This gives:

$M \cdot \ddot{\zeta}(t) + S \cdot \zeta(t) = F(t)$, where $M \leftrightarrow$ mass matrix

$S \leftrightarrow$ stiffness matrix

$F \leftrightarrow$ "load vector"

(iv) To use a numerical method, we first rewrite

the above as a system of first order ODE:

$$\begin{cases} M\dot{\zeta}(t) = M\eta(t) \\ M\dot{\eta}(t) + S\zeta(t) = F(t) \end{cases}$$

(see exact sol., $u_t = v$)

Next, we apply CN to each of these ODE:

$$M\left(\frac{\zeta^{(n+1)} - \zeta^{(n)}}{k}\right) = M\left(\frac{\eta^{(n+1)} + \eta^{(n)}}{2}\right)$$

$$M\left(\frac{\eta^{(n+1)} - \eta^{(n)}}{k}\right) + S\left(\frac{\zeta^{(n+1)} + \zeta^{(n)}}{2}\right) = \frac{F(t_{n+1}) - F(t_n)}{2}$$, where

k is the stepsize and $t_n = n \cdot k$.

This system can be written in block matrix form:

$$\begin{pmatrix} M & -\frac{k}{2}M \\ \frac{k}{2}S' & M \end{pmatrix}\begin{pmatrix} \zeta^{(n+1)} \\ \eta^{(n+1)} \end{pmatrix} = \begin{pmatrix} M & \frac{k}{2}M \\ -\frac{k}{2}S' & M \end{pmatrix}\begin{pmatrix} \zeta^{(n)} \\ \eta^{(n)} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{k}{2}\left(F(t_{n+1}) + F(t_n)\right) \end{pmatrix}$$

Th. When $f \equiv 0$, the CN scheme preserves the energy!!
(Backward) Euler not!

Chapter VIII: Laplace transforms

Goal: Introduce and study a tool for solving particular DE

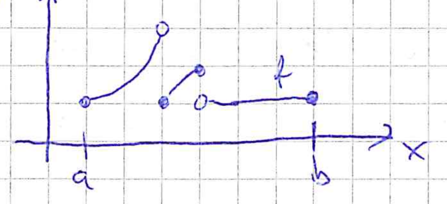
Use now "Lecture notes in Fourier analysis"

1) Preliminaries:

Def. A fct. $f: [a,b] \to \mathbb{R}$ is piecewise continuous, if the number of discontinuous points is finite, and its left and right limits at the discontinuity points exist.

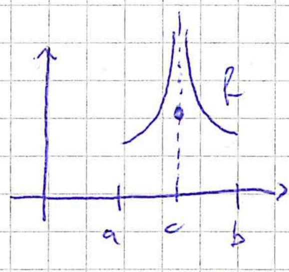
(5) Ex: • Any continuous f is pw continuous :-)

• Graph of a pw cont fct



finite nbr of breaks
no blowup to $\pm\infty$

Graph of a fct that is not pwc



$\lim_{t \to c^+} f(t) = \lim_{t \to c^-} f(t) = \infty$

Def. A fct $f: [0,\infty[\to \mathbb{R}$ is of exponential order if, $\exists$ constants $M, a, T > 0$ s.t. $|f(t)| \leq M e^{at}$ $\forall t \geq T$