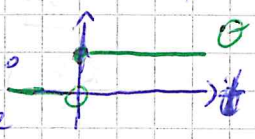
(S) Ex: • All nice fct are of exp. order: $f(t)=\sin(t)$, $f(t)=t^{22}$, $f(t)=5e^{2t}$, etc.

• $f(t)=e^{t^2}$ is not of exp. order

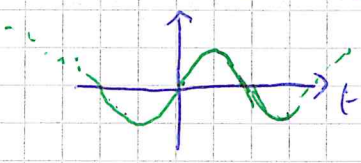
Def: A fct f is called causal if $f(t)=0 \ \forall t<0$.

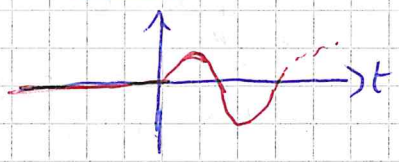
An easy way to make any fct causal is to multiply it with the Heaviside fct

$$\Theta(t)=\begin{cases}1 & t\geq 0\\ 0 & \text{else}\end{cases}$$



Ex: Draw $f(t)=\sin(t)$ and $\Theta(t)f(t)$ on $(-\infty,\infty)$



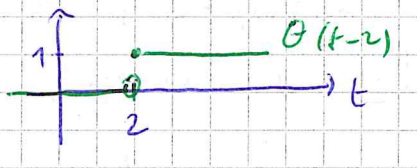


Recall: One can shift a fct to the left $f(t+T)$

to the right $f(t-T)$

Ex: Shift $\Theta(t)$ to the right by 2.

(S) $$\Theta(t-2)=\begin{cases}1 & \text{if } t\geq 2\\ 0 & \text{if } t<2\end{cases}$$



2) Laplace transform:

(Obs: causal ≥ 0)

Def: The Laplace transform (LT) of $f:[0,\infty)\to\mathbb{R}$ is defined by

$$\mathcal{L}\{f\}(s)=F(s)=\int_0^\infty f(t)e^{-st}\,dt$$ when this integral is finite (domain of F !!)

Ex: • $\mathcal{L}\{\Theta\}(s) \overset{\text{Def LT}}{=} \int_0^\infty \Theta(t)e^{-st}\,dt \overset{\text{Def }\Theta}{=} \int_0^\infty 1\cdot e^{-st}\,dt \overset{\text{Def }\infty}{=} \lim_{T\to\infty}\int_0^T e^{-st}\,dt =$

(S) $$= \lim_{T\to\infty}\left.\frac{e^{-st}}{-s}\right|_{t=0}^{T} = \lim_{T\to\infty}\left(\frac{e^{-sT}-1}{-s}\right) \overset{s>0}{=} \left(\frac{-1}{s}\right)\cdot(0-1)=\frac{1}{s}$$ //

only defined for $s>0$!!

Hence, $\mathcal{L}\{\Theta\}(s)=\frac{1}{s}$ for $s>0$ //

• Let $c\in\mathbb{R}$, $\mathcal{L}\{e^{ct}\}(s)=\int_0^\infty e^{ct}e^{-st}\,dt=\int_0^\infty e^{(c-s)t}\,dt =$

$$= \frac{1}{c-s}\left(\lim_{T\to\infty}\left(e^{(c-s)T}-1\right)\right) = \frac{1}{c-s}\cdot(0-1)=\frac{1}{s-c} \Rightarrow$$

↑ only def. for $c-s<0$ or $s>c$!!

$\Rightarrow \mathcal{L}\{e^{ct}\}(s) = \frac{1}{s-c}$ for $s>c$ //

Th: Assume that $f(t)$, for $t \geq 0$, is PW continuous and of exponential order a, for some $a \geq 0$. Then, the LT $\mathcal{L}\{f\}(s)$ ∃.

Proof: Def LT $\mathcal{L}\{f\}(s) = \int_0^\infty f(t) e^{-st}\,dt \overset{x \leq |x|}{\leq} \int_0^\infty |f(t)| e^{-st}\,dt \overset{|f(t)| \leq M e^{at}}{\leq}$

$\leq M \int_0^\infty e^{at} e^{-st}\,dt = \frac{M}{s-a} < \infty$ for $a-s<0$

"0" ↑ Split ... $\int_0^T + \int_T^\infty \ldots$ previous expl.

Properties of LT:

Th: LT is a linear operator: For $a,b \in \mathbb{R}$, and f, g, one has

$\mathcal{L}\{af+bg\}(s) = a\,\mathcal{L}\{f\}(s) + b\,\mathcal{L}\{g\}(s)$ (if all LT ∃)

Proof: Use linearity of integrals ...

Ex: $\mathcal{L}\{4+3e^{-2t}\}(s) = ?$ **Hint:** $4 = 4 \cdot \theta(t)$ for $t \geq 0$.

(5) $\mathcal{L}\{4+3e^{-2t}\}(s) \overset{\text{Th.}}{=} \mathcal{L}\{4\theta(t)\}(s) + 3 \cdot \mathcal{L}\{e^{-2t}\}(s) \overset{\text{above expl.}}{=} 4 \cdot \frac{1}{s} + 3 \cdot \frac{1}{s-(-2)} =$

$= \frac{4}{s} + \frac{3}{s+2}$ for $s>0$ and $s>-2$ (i.e. $s>0$)

Ex: Compute $\mathcal{L}\{\sin(at)\}(s)$ and $\mathcal{L}\{\cos(at)\}(s)$, where $a \in \mathbb{R}$

Hint: $e^{i\alpha} = \cos(\alpha) + i\sin(\alpha)$

$\mathcal{L}\{e^{iat}\}(s) \overset{\text{above expl}}{=} \frac{1}{s-ia} = \frac{1}{s-ia} \cdot \frac{s+ia}{s+ia} \overset{i^2=-1}{=} \frac{s+ia}{s^2+a^2} =$

$= \frac{s}{s^2+a^2} + i\,\frac{a}{a^2+s^2}$

$\mathcal{L}\{e^{iat}\}(s) = \mathcal{L}\{\cos(at)\}(s) + i\,\mathcal{L}\{\sin(at)\}(s)$

↳ $\mathcal{L}\{\cos(at)\}(s) = \frac{s}{s^2+a^2}$ and $\mathcal{L}\{\sin(at)\}(s) = \frac{a}{s^2+a^2}$ $(s>0)$

Th: (shifting 1)

If f has the LT $F(s) = \mathcal{L}\{f\}(s)$ and $c \in \mathbb{R}$, then

$$\mathcal{L}\{e^{ct} f(t)\}(s) = F(s-c) \quad \text{for } s > c.$$

Proof: Use def of LT to get

$$\mathcal{L}\{e^{ct} f(t)\}(s) \overset{\text{Def LT}}{=} \int_0^\infty e^{ct} f(t) e^{-st}\, dt = \int_0^\infty f(t) e^{-(s-c)t}\, dt \overset{\text{Def LT}}{=} F(t-s)$$

■

Ex: Compute LT $\mathcal{L}\{3 \cdot e^{-2t} \cos(8t)\}(s)$:

$$\mathcal{L}\{3e^{-2t}\cos(8t)\}(s) \overset{\text{linearity}}{=} 3\,\mathcal{L}\{e^{-2t}\cos(8t)\}(s) \overset{\text{shift w/ } c=-2}{=} 3\,\frac{s-(-2)}{(s-(-2))^2+8^2} =$$

$$\mathcal{L}\{\cos(8t)\}(s) = \frac{s}{s^2+8^2}$$

$$= \frac{3(s+2)}{(s+2)^2+64} \quad //$$

Th: (shifting 2)

If $f(t-T)$ is zero for $t \le T$, then

$$\mathcal{L}\{f(t-T)\}(s) = e^{-Ts} F(s) \quad \text{for } s > 0.$$

Proof:

$$\mathcal{L}\{f(t-T)\}(s) \overset{\text{Def LT}}{=} \int_0^\infty f(t-T) e^{-st}\, dt \overset{\text{Def } f_T}{=} \int_T^\infty f(t-T) e^{-st}\, dt =$$

$d\tau = dt$, $\tau = t-T$

$$= \int_0^\infty f(\tau) e^{-s(\tau+T)}\, d\tau = e^{-sT} \int_0^\infty f(\tau) e^{-s\tau}\, d\tau \overset{\text{Def LT}}{=}$$

$$= e^{-sT} F(s)$$

■

Rem: If $f(t-T)$ is not zero $\rightsquigarrow$ use Heaviside fct θ:

$$\mathcal{L}\{\theta(t-T) f(t-T)\}(s) = e^{-Ts} F(s).$$

Ex:

$$\mathcal{L}\{\theta(t-T)\}(s) \overset{\text{shift}}{=} e^{-Ts}\,\mathcal{L}\{\theta\}(s) \overset{\text{Table}}{=} e^{-Ts} \cdot \frac{1}{s}$$

Th: (Derivative of LT)

Let F be the LT of f. If F' ∃, one has

$$\mathcal{L}\{t f(t)\}(s) = -F'(s)$$

And similarly: $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n F^{(n)}(s)$ (+ assumptions)

Proof:

$$F'(s) \overset{\text{Def } F}{=} \frac{d}{ds}\int_0^\infty f(t)\,e^{-st}\,dt \overset{\text{Leibniz rule}}{=} \int_0^\infty f(t)\,\frac{d}{ds}e^{-st}\,dt = \int_0^\infty (-t)\,f(t)\,e^{-st}\,dt =$$

$$= -\mathcal{L}\{t f(t)\}(s)$$

$\uparrow$ Def LT

Ex: Compute $\mathcal{L}\{t e^{2t}\}(s) = ?$

We know $\mathcal{L}\{e^{2t}\}(s) = \frac{1}{s-2}$

The theorem gives $\mathcal{L}\{t e^{2t}\}(s) = -\left(\frac{1}{s-2}\right)' = \frac{1}{(s-2)^2}$ //

Ex: Compute $\mathcal{L}\{t\sinh(t)\}(s) = ?$

We know $\mathcal{L}\{\sinh(t)\}(s) = \frac{1}{s^2-1}$ (Table or using $\sinh(t) = \frac{1}{2}(e^t - e^{-t})$)

Theorem gives $\mathcal{L}\{t\sinh(t)\}(s) = -\left(\frac{1}{s^2-1}\right)' = \frac{2s}{(s^2-1)^2}$ //

Th: (Integral of LT)

Let $F(s)$ denote LT of f and assume $\lim_{t\to 0}\frac{1}{t}f(t)$ ∃.

Then, $\mathcal{L}\{\frac{1}{t}f(t)\}(s) = \int_s^\infty F(\omega)\,d\omega$

Proof: Def LT ↓

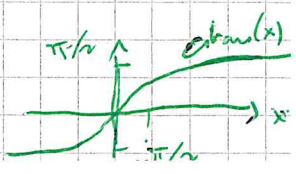
$$\int_s^\infty F(\omega)\,d\omega \overset{\text{Def LT}}{=} \int_s^\infty \int_0^\infty f(t)\,e^{-\omega t}\,dt\,d\omega \overset{\text{Fubini}}{=} \int_0^\infty f(t)\int_s^\infty e^{-\omega t}\,d\omega\,dt =$$

$$= \int_0^\infty f(t)\left(\frac{e^{-\omega t}}{-t}\Big|_{\omega=s}^{\infty}\right)dt \overset{t>0}{=} \int_0^\infty f(t)\left(0 + \frac{e^{-st}}{t}\right)dt = \int_0^\infty \frac{f(t)}{t}e^{-st}\,dt =$$

$$= \mathcal{L}\{\frac{1}{t}f(t)\}(s)$$

$\uparrow$ Def LT, ∃ limit

Ex: Compute LT $\mathcal{L}\{\frac{\sin(t)}{t}\}(s)$

- $f(t) = \sin(t)$ and $F(s) = \mathcal{L}\{f\}(s) = \frac{1}{s^2+1}$ (table) and $\lim_{t\to 0}\frac{1}{t}f(t) = 1$
- Theorem gives

$$\mathcal{L}\{t\sin(t)\}(s) = \int_s^\infty \frac{d\omega}{\omega^2+1} = \arctan(\omega)\Big|_s^\infty = \frac{\pi}{2} - \operatorname{atan}(s)$$