⚠ **Th:** (LT of derivatives)

Let $F(s)$ denote the LT of f and assume that f' has a LT. Then,

$$\mathcal{L}\{f'\}(s) = s\,F(s) - f(0) \quad \text{(for } s>0 \text{ or } s>a \text{ if } f' \text{ of exp. order } a\text{)}$$

Proof:

$$\mathcal{L}\{f'\}(s) \overset{\text{Def LT}}{=} \int_0^\infty f'(t)e^{-st}\,dt \overset{\text{int. by part}}{=} f(t)e^{-st}\Big|_{t=0}^{\infty} - \int_0^\infty f(t)(-s)e^{-st}\,dt =$$

$$\underset{s>0}{=} 0 - f(0) + s\int_0^\infty f(t)e^{-st}\,dt \underset{\text{Def LT}}{=} -f(0) + sF(s)$$

∎

Rem: Similarly, one has

$$\mathcal{L}\{f^{(n)}\}(s) = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0)$$

(under app. assumptions)

Th. (LT of integral)

Let $F(s)$ denote the LT of f. Then,

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\}(s) = \frac{F(s)}{s}.$$

Proof:

- Set $h(t) := \int_0^t f(\tau)\,d\tau$ and observe $h'(t) = f(t)$ and $h(0) = 0$. (*)
- The above Th. gives $\underbrace{\mathcal{L}\{h'\}(s)}_{\mathcal{L}\{f\}(s) = F(s)} = \underbrace{s\cdot\mathcal{L}\{h\}(s) - h(0)}_{s\cdot\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}(s) \ (*)}$

Hence, $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\}(s) = \dfrac{F(s)}{s}$

∎

3) The inverse Laplace transform:

Def: The inverse Laplace transform (ILT) of $F(s)$ is the (pwc and of exp. order) fct. $f(t)$ s.t. $\mathcal{L}\{f\}(s) = F(s)$.

Notation: $f(t) = \mathcal{L}^{-1}\{F(s)\}(t)$.

Rem: ILT is usually done using tables of LT.

- Since $\mathcal{L}$ is a linear operator, so is $\mathcal{L}^{-1}$:

$$\mathcal{L}^{-1}\{aF(s)+bG(s)\}(t) = a\,\mathcal{L}^{-1}\{F(s)\}(t) + b\,\mathcal{L}^{-1}\{G(s)\}(t) = a f(t) + b g(t)$$

$\forall a, b, F, G.$

Ex: Find ILT of $F(s) = \frac{2}{s} - \frac{3}{s^2} + \frac{4s}{s^2+1}$

One has $f(t) = \mathcal{L}^{-1}\{F(s)\}(t) \overset{\text{linearity}}{=} 2\cdot\mathcal{L}^{-1}\{\frac{1}{s}\}(t) - 3\,\mathcal{L}^{-1}\{\frac{1}{s^2}\}(t) + 4\,\mathcal{L}^{-1}\{\frac{s}{s^2+1}\}(t) \underset{\text{Table}}{=}$

$$= 2\Theta(t) - 3t + 4\cos(t)$$

Method of partial fractions:

Prob: Compute $\mathcal{L}^{-1}\{F(s)\}(t)$, when $F(s) = \frac{Q(s)}{P(s)}$, where P, Q are real polynomials with $\deg(Q) < \deg(P)$.

Idea: Decompose F into partial fractions.

Consider 3 cases:

(i) P is a quadratic polyn. (of degree 2) with 2 distinct roots:

$$F(s) = \frac{Q(s)}{P(s)} = \frac{2s-8}{s^2-5s+6} \underset{\substack{\text{roots}\\ 2,3}}{=} \frac{2s-8}{(s-2)(s-3)} \underset{\substack{\text{We want}\\ \text{this decomposition.}}}{\overset{!}{=}} \frac{A}{s-2} + \frac{B}{s-3}$$

$$= \frac{A(s-3)+B(s-2)}{(s-2)(s-3)} = \underbrace{\frac{s(A+B) + (-3A-2B)}{(s-2)(s-3)}}_{\text{simpler}}$$

Compare coeff: s^0: $-8 \overset{!}{=} (-3A-2B)$
s^1: $2 \overset{!}{=} (A+B)$ $\Big\} \Rightarrow A = 4$ and $B = -2$.

Hence, $F(s) = \frac{4}{s-2} + \frac{(-2)}{s-3}$

ILT is then given by:

$$f(t) = \mathcal{L}^{-1}\{F(s)\}(t) \overset{\text{linearity}}{=} 4\,\mathcal{L}^{-1}\{\frac{1}{s-2}\}(t) - 2\,\mathcal{L}^{-1}\{\frac{1}{s-3}\}(t) \overset{\text{Table}}{=} 4e^{2t} - 2e^{3t}$$

(ii) $P(s)$ is a quadratic polyn. with a double root:

(5)

$$F(s) = \frac{Q(s)}{P(s)} = \frac{s+1}{(s+2)^2}$$, Root of P is -2.

Write $$\frac{s+1}{(s+2)^2} \overset{!}{=} \frac{A}{(s+2)} + \frac{B}{(s+2)^2} = \frac{As+(2A+B)}{(s+2)^2}.$$

Compare coeff: s^0: $1 \overset{!}{=} 2A+B$
s^1: $1 \overset{!}{=} A$
$\Rightarrow A=1,\ B=-1.$

Hence, $$F(s) = \frac{1}{s+2} + \frac{(-1)}{(s+2)^2}$$ and finally

$$f(t) = \mathcal{L}^{-1}\{F(s)\}(t) \underset{\text{linearity}}{=} \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}(t) \underset{\text{Table}}{=} e^{-2t} - e^{-2t}\cdot t$$

(iii) $P(s)$ is a quadratic polyn. w/ complex roots:

$$F(s) = \frac{Q(s)}{P(s)} = \frac{s+1}{s^2+4s+5} = \frac{s+1}{(s+2)^2+1} = \frac{s+1+1-1}{(s+2)^2+1} =$$

Complete square: $s^2+4s+4+1$

$$= \frac{s+2}{(s+2)^2+1} - \frac{1}{(s+2)^2+1}.$$

Hence, $$f(t) = \mathcal{L}^{-1}\{F(s)\}(t) \underset{\text{Linearity}}{=} \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}(t) \underset{\text{Table}}{=}$$

$$= e^{-2t}\cos(t) - e^{-2t}\sin(t)$$

Ex: Compute $$\mathcal{L}^{-1}\Bigg\{\underbrace{\frac{s+2}{s^3-s^2+s-1}}_{F(s)}\Bigg\}(t)$$

- Obs: $s^3-s^2+s-1 = s^2(s-1)+(s-1) = (s^2+1)(s-1)$

(5)

- Decompose $$F(s) \overset{!}{=} \underbrace{\frac{A}{s-1}}_{(i)} + \underbrace{\frac{Bs+C}{(s^2+1)}}_{(iii)\text{ above}} = \frac{(A+B)s^2+(C-B)s+(A-C)}{(s-1)(s^2+1)}$$

- Compare coeff: s^0, s^1, s^2: $$\begin{cases} 2 = A-C \\ 1 = C-B \\ 0 = A+B \end{cases} \Rightarrow A=3/2,\ B=-3/2,\ C=-1/2$$

- Finally: $\mathcal{L}^{-1}\{F(s)\}(t) \underset{\text{linearity}}{=} \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}(t) - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t) \underset{\text{Table}}{=}$

$$= \frac{3}{2}e^{t} - \frac{3}{2}\cos(t) - \frac{1}{2}\sin(t) \;//$$

4) Applications of Laplace transforms:

Goal: Solve IVP and integral equations

Idea: (i) Take LT of the prob. ⇝ get algebraic eq.

(ii) Solve alg. eq.

(iii) Take ILT to find exact sol. to the problem.

Ex. (Solving IVP with LT)

Problem $\begin{cases} y'(t) + 2y(t) = 12e^{3t} \\ y(0) = 3 \end{cases}$

(i) Take LT: $\mathcal{L}\{y'(t) + 2y(t)\}(s) = 12\mathcal{L}\{e^{3t}\}(s)$ or

S $\quad s\,\mathcal{L}\{y(t)\}(s) - y(0) + 2\mathcal{L}\{y(t)\}(s) = 12\,\frac{1}{s-3}$

(ii) Set $Y(s) := \mathcal{L}\{y(t)\}(s)$ and get the algebraic eq.

$$sY(s) - y(0) + 2Y(s) = \frac{12}{s-3} \quad \text{or} \quad Y(s) = \frac{3(1+s)}{(s-3)(s+2)}$$

(iii) Take ILT to get back $y(t)$:

S $\quad y(t) = \mathcal{L}^{-1}\{Y(s)\}(t) = \underbrace{\mathcal{L}^{-1}\left\{\frac{3(1+s)}{(s-3)(s+2)}\right\}(t)}_{\text{use partial fractions to compute this ILT!}}$

$$\frac{3(1+s)}{(s-3)(s+2)} \underset{\uparrow}{\overset{!}{=}} \frac{A}{s-3} + \frac{B}{s+2} = \frac{A(s+2) + B(s-3)}{(s-3)(s+2)} = \frac{(A+B)s + (-3B+2A)}{(s-3)(s+2)}$$

2 distinct roots: -2 and 3

Compare coeff (s^0, s^1) and get $A = 3/5$ and $B = 12/5$

Hence: $Y(s) = \frac{3}{5}\,\frac{1}{s-3} + \frac{12}{5}\,\frac{1}{s+2}$ and

$y(t) = \mathcal{L}^{-1}\{Y(s)\}(t) = \frac{3}{5}e^{-3t} + \frac{12}{5}e^{2t} \;//$ ⚠ Can verify that this is a sol. to IVP ⚠