Ex: (Solving IVP with LT)

Problem $\begin{cases} y''(t) + 4y'(t) + 3y(t) = 0 \\ y(0) = 3,\ y'(0) = 1 \end{cases}$

(i) Take LT: $\mathcal{L}\{y''(t)\}(s) + 4\mathcal{L}\{y'(t)\}(s) + 3\mathcal{L}\{y(t)\}(s) = 0 \overset{\text{Table}}{\underset{Y(s) = \mathcal{L}\{y\}(s)}{\Longleftrightarrow}}$

(S) $s^2 Y(s) - s\underbrace{y(0)}_{=3} - \underbrace{y'(0)}_{=1} + 4(sY(s) - \underbrace{y(0)}_{=3}) + 3Y(s) = 0$

(ii) Find $Y(s)$: $Y(s) = \dfrac{3s+13}{s^2+4s+3} = \dfrac{3s+13}{(s+3)(s+1)} \underset{\text{part. fract.}}{\overset{!}{=}} \dfrac{A}{s+3} + \dfrac{B}{s+1}$

... Get $A = -2$ and $B = 5$

Hence, $Y(s) = -\dfrac{2}{s+3} + \dfrac{5}{s+1}$

(S) (iii) Take ILT: $y(t) = \mathcal{L}^{-1}\{Y(s)\}(t) \underset{\text{Table}}{=} -2e^{-3t} + 5e^{-t}$ //

Ex: (Solve integral eq. using LT)

Problem (see .pdf): Current $i(t)$ in electric circuit given by

$$\begin{cases} i(t) + \int_0^t i(\tau)\,d\tau = v(t) \\ i(0) = 0, \end{cases}$$

where $v(t) = \theta(t-1) - \theta(t-2)$

Find $i(t)$!

(i) Take LT: $\mathcal{L}\{i(t)\}(s) + \mathcal{L}\left\{\int_0^t i(\tau)\,d\tau\right\}(s) = \mathcal{L}\{\theta(t-1) - \theta(t-2)\}(s)$.

(ii) Set $I(s) := \mathcal{L}\{i(t)\}(s)$ and use properties of LT to get:

$$I(s) + \frac{I(s)}{s} = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} \qquad = \left(\frac{1+s}{s}\right) I(s)$$

Hence, $I(s) = \dfrac{e^{-s}}{s+1} - \dfrac{e^{-2s}}{s+1}$

(iii) Take ILT to get back $i(t)$:

$$i(t) = \mathcal{L}^{-1}\{I(s)\}(t) = e^{-(t-1)}\,\theta(t-1) - e^{-(t-2)}\,\theta(t-2)$$

<u>Obs</u>: Using def of θ, one gets $i(t) = \begin{cases} 0 & , t<1 \\ e^{-(t-1)} & , 1<t<2 \\ e^{-(t-1)} - e^{-(t-2)} & , 2<t \end{cases}$

5) Convolutions:

See .pdf on canvas

Chapter IX: Fourier analysis

<u>Goal</u>: Study approx. of fct by "simple" trigonometric fct.

<u>Applications</u>: signal processing (mp3), jpeg compression, etc.

1) Periodic functions:

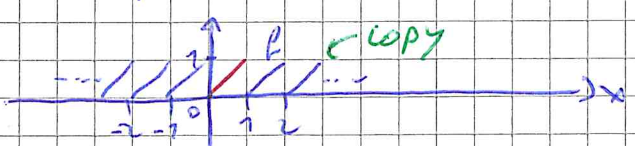
<u>Def</u>: • A fct f is called <u>periodic</u> if $\exists\, p>0$ s.t.

$$f(x+p) = f(x) \quad \forall x \in \mathbb{R}$$

• The smallest such p is called the (prime) <u>period</u>

<u>Ex</u>: The period of $\sin(x)$ or $\cos(x)$ is $p = 2\pi$.

<u>Ex</u>: Consider x for $0<x<1$ and extend it periodically $\rightsquigarrow f(x)$



f has period $p=1$

<u>Lemma</u>: Let f be integrable and periodic of period p.

Then, the integral $\int_a^{a+p} f(x)\,dx$ does <u>not</u> depend on the starting point a.

Proof:

WLOG $a < p$: $\int_a^{a+p} f(x)dx = \int_a^p f(x)dx + \int_p^{a+p} f(x)dx =$ $\quad y = x-p$, $dy = dx$

$$= \int_a^p f(x)dx + \int_0^a \underbrace{f(y+p)}_{=f(y)\ (\text{periodic})} dy = \int_0^p f(x)dx$$

S

$a \geq p \to$ same

2) Fourier series:

Def. Let f be 2π-periodic and integrable over $[-\pi, \pi]$.

The expression

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$$

$\cos(0 \cdot x)$

don't know if series converge!

is called the Fourier series of f (FS), where the Fourier coefficients are

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)dx \quad \text{for } n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)dx \quad \text{for } n = 1, 2, \ldots$$

Rem: by above Lemma, can integrate $\int_0^{2\pi}$ or any interval of length 2π.

In order to get a more compact form for (FS), use

$e^{inx} = \cos(nx) + i\sin(nx)$

$e^{-inx} = \cos(nx) - i\sin(nx)$

$\Rightarrow \cos(nx) = \dfrac{e^{inx} + e^{-inx}}{2}$ and $\sin(nx) = \dfrac{e^{inx} - e^{-inx}}{2i}$

Hence, one gets

$$f(x) \sim \frac{a_0}{2} + \sum_{n=0}^{\infty} \left\{ a_n \left(\frac{e^{inx} + e^{-inx}}{2} \right) + b_n \left(\frac{e^{inx} - e^{-inx}}{2i} \right) \right\} = \ldots =$$

$$= \underbrace{\frac{a_0}{2}}_{=:c_0} + \sum_{n=1}^{\infty} \underbrace{\left(\frac{a_n - ib_n}{2} \right)}_{=:c_n} e^{inx} + \sum_{n=1}^{\infty} \underbrace{\left(\frac{a_n + ib_n}{2} \right)}_{=:c_{-n}} e^{-inx}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{inx} \leftarrow \text{complex Fourier series of } f \text{ (FS)}$$

Rem: $c_0 = \frac{a_0}{2}$, $c_n = \frac{a_n - i b_n}{2}$, $c_{-n} = \overline{c_n}$ and $a_0 = 2c_0$, $a_n = c_n + c_{-n}$

$b_n = i(c_n - c_{-n})$

Rem: Recall that fct and FEM $u(x) = \sum_{j=1}^{n} c_j \varphi_j(x)$:-)

3) Justification of the above:

Lemma: $\{e^{inx}\}_{n=-\infty}^{\infty}$ is an orthogonal set on $[-\pi, \pi]$, that is

$$\int_{-\pi}^{\pi} e^{inx} \cdot e^{-ikx}\, dx = \begin{cases} 0 & , n \neq k \\ 2\pi & , n = k \end{cases} \qquad \forall\, n, k \in \mathbb{Z}$$

Rem: This is L^2-inner prod.! e^{inx} $\leftarrow$ "basis" fct in $L^2(-\pi,\pi)$

Proof:

- If $n = k$: $\int_{-\pi}^{\pi} e^{inx} e^{-ikx}\, dx = \int_{-\pi}^{\pi} 1\, dx = 2\pi$:-)
- If $n \neq k$: $\int_{-\pi}^{\pi} e^{i(n-k)x}\, dx = \frac{e^{i(n-k)x}}{i(n-k)}\Big|_{x=-\pi}^{\pi} = \frac{\cos((n-k)x) + i\sin((n-k)x)}{i(n-k)}\Big|_{x=-\pi}^{\pi} =$

$$= \frac{(-1)^{n-k} - (-1)^{n-k}}{i(n-k)} = 0 \quad :-)$$

The above is used to compute the Fourier coeff. c_n:

- Multiply FS $f(x) \underset{\uparrow \text{ assume}}{=} \sum_{n=-\infty}^{\infty} c_n e^{inx}$ by e^{-ikx}, for some fixed k, integrate

$$\int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx = \sum_{n=-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{i(n-k)x}\, dx \overset{\text{lemma}}{=} c_k \cdot 2\pi$$

- This gives $c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx$!
- This also gives formulas for a_0, a_n, b_n:

$$a_0 = 2 \cdot c_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx \,!$$

$$a_n = c_n + c_{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \underbrace{\left(e^{-inx} + e^{inx}\right)}_{2\cdot\cos(nx)} dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

and similarly for b_n ...