Chapter 8: Laplace transform (summary)

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Goal: Introduce and study a tool to find the exact solution to particular DEs and integral equations.

- A function $f\colon [a,b] \to \mathbb{R}$ is piecewise continuous if its number of discontinuous points is finite and its left and right limits at the discontinuous points exist.
- A function f is of exponential order α if there exists positive constants T and M such that

  $$|f(t)| \leq M\mathrm{e}^{\alpha t} \quad \text{for all } t \geq T.$$

  All the nice functions $\sin(3t), \mathrm{e}^{5t}, t^4+5t^2+23, \ldots$ are of exponential order α, for some α. The function e^{t^2} is an example of a function that is not of exponential order α for any α.
- A function f is called causal if $f(t) = 0$ for $t < 0$.
- The Heaviside function, or unit step function, is defined by

  $$\theta(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0. \end{cases}$$

- The Laplace transform (LT) of a function $f\colon [0,\infty) \to \mathbb{R}$ is the function F defined by the integral

  $$F(s) := \mathscr{L}\{f(t)\}(s) := \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t.$$

  The domain of definition of the Laplace transform is $D(F) = \{s \in \mathbb{R} : \text{the above integral exists}\}$.

  The Laplace transform of a piecewise continuous function that is of exponential order α exists for $s > \alpha$.

  Linearity of LT: If f, f_1, f_2 are functions whose Laplace transforms exist for $s > \alpha$ and c is a real constant, then the following holds for $s > \alpha$:

  $$\mathscr{L}\{f_1(t) + f_2(t)\}(s) = \mathscr{L}\{f_1(t)\}(s) + \mathscr{L}\{f_2(t)\}(s)$$
  $$\mathscr{L}\{cf(t)\}(s) = c\mathscr{L}\{f(t)\}(s).$$

- We have the following properties of the Laplace transform $F(s) = \mathscr{L}\{f(t)\}(s)$:

  $$\mathscr{L}\{\mathrm{e}^{ct} f(t)\}(s) = F(s-c) \quad \text{for } s > c.$$
  $$\mathscr{L}\{f(t-T)\theta(t-T)\}(s) = \mathrm{e}^{-Ts}F(s) \quad \text{for } s > 0.$$
  $$\mathscr{L}\{t^n f(t)\}(s) = (-1)^n \frac{\mathrm{d}^n F}{\mathrm{d}s^n}(s).$$
  $$\mathscr{L}\{\frac{1}{t} f(t)\}(s) = \int_s^\infty F(\omega)\,\mathrm{d}\omega \quad \text{if} \quad \lim_{t\to 0} \frac{f(t)}{t} \quad \text{exists.}$$
  $$\mathscr{L}\{f^{(n)}\}(s) = s^n \mathscr{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0).$$
  $$\mathscr{L}\left\{\int_0^t f(\tau)\,\mathrm{d}\tau\right\} = \frac{F(s)}{s}.$$

  The above and Laplace transforms to usual functions are found in tables of Laplace transforms.

- Given a function $F(s)$, if there is a (piecewise continuous and of exponential order) function $f(t)$ on $[0,\infty)$ which satisfies
$$\mathscr{L}\{f\} = F,$$
then f is called the inverse Laplace transform of F (ILT) and it is denoted by $f = \mathscr{L}^{-1}\{F\}$.

  Linearity of ILT: As before, we have the following rules
$$\mathscr{L}^{-1}\{F_1 + F_2\} = \mathscr{L}^{-1}\{F_1\} + \mathscr{L}^{-1}\{F_2\}$$
$$\mathscr{L}^{-1}\{cF\} = c\mathscr{L}^{-1}\{F\}.$$

- The method of partial fractions permits to break rational functions $F(s) = \frac{Q(s)}{P(s)}$, where $\deg(Q) < \deg(P)$, into smaller and easier parts. This is then used to find the inverse Laplace transform of $F(s)$. We have seen the following examples (a bit more general than in the lecture)

  i Nonrepeated linear factors. Determine A, B and C such that
$$\frac{s^2+2}{(s-1)(s-2)(s+1)} \stackrel{!}{=} \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+1}.$$

  ii Repeated linear factors. Determine A, B, C and D such that
$$\frac{4s+8}{(s-2)^2(s+2)^2} \stackrel{!}{=} \frac{A}{(s-2)^2} + \frac{B}{(s-2)} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)}.$$

  iii Quadratic factors. Determine A, B and C such that
$$\frac{8s^2+16}{(s-1)(s^2+2s+5)} = \frac{8s^2+16}{(s-1)((s+1)^2+2^2)} \stackrel{!}{=} \frac{A}{(s-1)} + \frac{B(s+1)+2C}{((s+1)^2+2^2)}.$$

- We can use the Laplace transform to solve IVP using the following recipe:

  i Take the Laplace transform of both sides of the differential equation.

  ii Use properties of the Laplace transform and the initial values of the IVP to solve an equation for the Laplace transform of the solution of the IVP.

  iii Take the inverse of the Laplace transform to obtain the solution of the IVP.

- Following the same recipe, one can use the Laplace transform to find exact solutions to integral equations, for instance
$$\begin{cases} i(t) + \int_0^t i(\tau)\,\mathrm{d}\tau = v(t) \\ i(0) = 0, \end{cases}$$
where $v(t) = \theta(t-1) - \theta(t-2)$.

- The convolution of two piecewise continuous functions f and g (defined on $[0,\infty)$) is a new function defined as
$$(f * g)(t) = \int_0^t f(t-v)g(v)\,\mathrm{d}v.$$
In connection with the Laplace transform, we have the following results (under usual hypothesis and definitions)
$$\mathscr{L}\{(f*g)(t)\}(s) = F(s)G(s)$$
$$\mathscr{L}^{-1}\{F(s)G(s)\}(t) = (f*g)(t).$$
This means that the inverse Laplace transform of a product of Laplace transforms is a convolution.

Further resources:

- math.lamar.edu
- khanacademy.org
- intmath.com
- ocw.mit.edu