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Chapter 9: Fourier analysis (summary)

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Goal: Study approximation of functions by simple trigonometric functions.
Applications: Signal processing, .mp3, .jpeg, etc.

- A function $f\colon \mathbb{R} \to \mathbb{C}$ such that there exists a $p > 0$ with $f(x + p) = f(x)$ for all $x \in \mathbb{R}$ is called p-periodic. The smallest such p is called the (prime) period of f.

- For an integrable p-periodic function f, the integral

$$\int_a^{a+p} f(x)\,\mathrm{d}x$$

does not depend on the point a.

- Let $f\colon \mathbb{R} \to \mathbb{C}$ be 2π-periodic and Riemann integrable on $[-\pi,\pi]$. The series

$$\sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}nx} \qquad \text{or} \qquad \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\big(a_n\cos(nx) + b_n\sin(nx)\big)$$

are called the Fourier series of f (FS), where

$$c_n := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{-\mathrm{i}nx}\,\mathrm{d}x \quad \text{for} \quad n \in \mathbb{Z},$$

and

$$a_0 := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x, \qquad a_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x, \qquad b_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x \quad \text{for} \quad n \geq 1,$$

are called the Fourier coefficients of f. One has the relations

$$a_0 = 2c_0, \quad a_n = c_n + c_{-n}, \quad b_n = \mathrm{i}(c_n - c_{-n}) \quad \text{and} \quad c_n = \frac{a_n - \mathrm{i}b_n}{2}, \quad c_{-n} = \bar{c}_n.$$

Observe that $b_n = 0$ if f is even (i. e. $f(-x) = f(x)$ for all x) and $a_n = 0$ if f is odd (i. e. $f(-x) = -f(x)$ for all x).

Observe also that one can integrate over any interval of length 2π since f (and cosine and sine) is 2π-periodic.

- The set $\{\mathrm{e}^{\mathrm{i}nx}\}_{n\in\mathbb{Z}}$ is an orthogonal set on $[-\pi,\pi]$, that is

$$\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}nx}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = \begin{cases} 0 & \text{if} \quad n \neq k \\ 2\pi & \text{else.} \end{cases}$$

- Bessel's inequality reads: Let f be 2π-periodic and square integrable, then

$$\sum_{n=-\infty}^{\infty} |c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,\mathrm{d}x.$$

In particular, this implies that the Fourier coefficients c_n go to zero as n goes to $\pm$ infinity (Riemann-Lebesgue lemma). A similar formula exists for the coefficients a_n and b_n (see compendium) which implies that a_n, b_n go to zero as n goes to infinity. These facts are needed, for example, to prove convergence results on the Fourier series (see below).

- We recall the following definitions. A function $f\colon [a,b] \to \mathbb{R}$ is piecewise continuous (notation $f \in PC([a,b])$) if f is continuous on $[a,b]$ except perhaps at finitely many points $x_1, x_2, \dots, x_n \in [a,b]$. At these points the left-hand and right-hand limits of f exist: $f(x_j-) = \lim_{h\to 0, h>0} f(x_j - h)$ and $f(x_j+) = \lim_{h\to 0, h>0} f(x_j + h)$. Similarly, f is piecewise smooth (notation $f \in PS([a,b])$) if $f, f' \in PC([a,b])$. Finally, $f \in PC(\mathbb{R})$, resp. $f \in PS(\mathbb{R})$, if f is piecewise continuous, resp. smooth, on every bounded interval $[a,b]$.

- Pointwise convergence of Fourier series: Consider f a 2π-periodic function and piecewise smooth on $\mathbb{R}$ (i. e. in $PS(\mathbb{R})$). Set $S_N^f(x) := \sum_{n=-N}^{N} c_n \mathrm{e}^{\mathrm{i}nx}$, where c_n are the Fourier coefficients of f. We have
$$\lim_{N\to\infty} S_N^f(x) = \frac{1}{2}\big(f(x-) + f(x+)\big) \quad \forall x \in \mathbb{R}.$$
In particular, if f is continuous at x, $\lim_{N\to\infty} S_N^f(x) = f(x)$ and we see that the Fourier series converges, in this case, to the value of $f(x)$!

- Parseval's identity reads: For $f \in \mathscr{L}^2(-\pi,\pi)$ a piecewise smooth 2π-periodic function, one has
$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 \,\mathrm{d}x.$$

- Derivative of Fourier series: Let f be a 2π-periodic, continuous and in $PS([-\pi,\pi])$, we have
$$c_n' = \mathrm{i}nc_n,$$
where c_n are the Fourier coefficients of f and c_n' those of f'. In terms of a_n and b_n, one has the relations $a_n' = nb_n$ and $b_n' = -na_n$.
With this in hand, one has the following result:
Let f be 2π-periodic, continuous, and piecewise smooth and suppose that f' is piecewise smooth. If $\sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}nx}$ is the Fourier series of $f(x)$, then $f'(x)$ is the derived series $\sum_{n=-\infty}^{\infty} \mathrm{i}nc_n \mathrm{e}^{\mathrm{i}nx}$ for all x at which $f'(x)$ exists. At jump points of f', the series converges to $\frac{1}{2}\big(f'(x-) + f'(x+)\big)$.

- Integral of Fourier series: Let f be 2π-periodic and in $PC(\mathbb{R})$ with Fourier coefficients c_n. Set $F(x) = \int_0^x f(y)\,\mathrm{d}y$. If $c_0 = 0$ then the Fourier coefficients of F are given by
$$C_n = \frac{c_n}{\mathrm{i}n} \quad \text{for} \quad n \neq 0$$
and $C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)\,\mathrm{d}x$. I. e. $F(x) = C_0 + \sum_{n\neq 0} \frac{c_n}{\mathrm{i}n}\mathrm{e}^{\mathrm{i}nx}$. If $c_0 \neq 0$, this series converges to $F(x) - c_0 x$. (This comes from the fact that the integral of a periodic function may not be periodic: $f(x) = 1$ is periodic but its integral $F(x) = x$ is not).

We now look at Fourier series of functions of arbitrary period.

- Using a simple change of variable, the Fourier series of a $2L$-periodic function f is given by
$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\Big(a_n \cos\Big(\frac{n\pi}{L}x\Big) + b_n \sin\Big(\frac{n\pi}{L}x\Big)\Big)$$
with the Fourier coefficients
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\Big(\frac{n\pi}{L}x\Big)\,\mathrm{d}x \quad \text{and} \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\Big(\frac{n\pi}{L}x\Big)\,\mathrm{d}x.$$

We now look at Fourier series of non-necessary periodic functions.

- Let f be defined on the interval $[0,\pi]$ (for ease of presentation, one can do the same, see lecture for an interval $[0,L]$) and integrable. Using the even extension of f on $[-\pi,\pi]$ defined by

$$f_{\text{even}}(-x) = f(x) \quad \text{for} \quad x \in [0,\pi] \quad \text{(observe that} \quad f_{\text{even}}(x) = f(x) \text{ for } x \in [0,\pi])$$

  one gets the Fourier cosine series of f

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

  with the coefficients

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) \, \mathrm{d}x.$$

  Using the odd extension of f on $[-\pi,\pi]$ defined by

$$f_{\text{odd}}(-x) = -f(x) \quad \text{for} \quad x \in (0,\pi] \quad \text{and} \quad f_{\text{odd}}(0) = 0$$

  one gets the Fourier sine series of f

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx)$$

  with the coefficients

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) \, \mathrm{d}x.$$

Further resources:

- wolfram.com (Fourier series)
- wikibooks.org (Fourier series)
- math.lamar.edu (periodic functions, orthogonal set)
- mathsisfun.com (Fourier series)
- khanacademy.org (Fourier series)
- intmath.com (Fourier series)