Chapter 5: FEM for two-point BVP (summary)

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Goal: We use the theoretical and practical tools from the previous sections to present and analyse FEM for several BVP.

- In order to get a FE approximation to the BVP ($a > 0$ and f are (nice and) given)

$$\begin{cases} -(a(x)u'(x))' = f(x) & \text{for} \quad x \in (0,1) \\ u(0) = 0 \quad \text{and} & u(1) = 0 \end{cases}$$

we proceed as usual:

1. Define the test/trial space $H_0^1 = \{v\colon [0,1] \to \mathbb{R} : \; v, v' \in L^2(0,1), v(0) = v(1) = 0\}$, multiply the DE with a test function $v \in H_0^1$, integrate over the domain $[0,1]$ and get the VF

$$\text{Find } u \in H_0^1 \quad \text{such that} \quad \int_0^1 a(x)u'(x)v'(x)\,\mathrm{d}x = \int_0^1 f(x)v(x)\,\mathrm{d}x \quad \forall v \in H_0^1.$$

Observe that the trial and test spaces are the same since the BVP has homogeneous Dirichlet BC.

2. Define the finite dimensional space $V_h^0 = \{v\colon [0,1] \to \mathbb{R} : \; v \text{ is cont. pw. linear on } T_h, v(0) = v(1) = 0\}$, where as usual T_h is a uniform partition with mesh $h = \frac{1}{m+1}$.
Observe that $V_h^0 = \text{span}(\varphi_1, \ldots, \varphi_m) \subset H_0^1$ with the hat functions φ_j.
The FE problem then reads

$$\text{Find } u_h \in V_h^0 \quad \text{such that} \quad \int_0^1 a(x)u_h'(x)\chi'(x)\,\mathrm{d}x = \int_0^1 f(x)\chi(x)\,\mathrm{d}x \quad \forall \chi \in V_h^0.$$

The above is also called $cG(1)$ FE (for linear continuous Galerkin FE).

3. Choosing $\chi = \varphi_i$ for $i = 1, \ldots, m$, writing $u_h(x) = \sum_{j=1}^m \zeta_j \varphi_j(x) \in V_h^0$, and inserting everything into the FE problem gives the following linear system of equations

$$S\zeta = b,$$

where the $m \times m$ stiffness matrix S has entries $s_{ij} = \int_0^1 a(x)\varphi_i'(x)\varphi_j'(x)\,\mathrm{d}x$ and the $m \times 1$ load vector b has entries $b_i = \int_0^1 f(x)\varphi_i(x)\,\mathrm{d}x$. Formulas for these entries can be found in the book. Solving this system gives the vector ζ and in turns the FE approximation u_h.

- The above needs minor adaptations when dealing with other BC.

Let us for example derive a FE approximation for the following BVP

$$\begin{cases} -u''(x) + 4u(x) = 0 & \text{for} \quad x \in (0,1) \\ u(0) = \alpha \quad \text{and} & u(1) = \beta, \end{cases}$$

where $\alpha \neq 0$ and $\beta \neq 0$ are given real number. Such boundary conditions are called non-homogeneous Dirichlet boundary conditions.

The derivation of a numerical approximation for solutions to the above problem is given by

1. Define the trial space $V = \{v\colon [0,1] \to \mathbb{R}:\ v, v' \in L^2(0,1), v(0) = \alpha, v(1) = \beta\}$ and the test space $V^0 = \{v\colon [0,1] \to \mathbb{R}:\ v, v' \in L^2(0,1), v(0) = v(1) = 0\}$. Multiply the DE with a test function $v \in V^0$, integrate over the domain $[0,1]$ and get the VF
$$\text{Find } u \in V \quad \text{such that} \quad \int_0^1 u'(x)v'(x)\,\mathrm{d}x + 4\int_0^1 u(x)v(x)\,\mathrm{d}x = 0 \quad \forall v \in V^0.$$
2. Next, define the finite dimensional spaces
$V_h = \{v\colon [0,1] \to \mathbb{R}:\ v \text{ is cont. pw. linear on } T_h \text{ and} v(0) = \alpha, v(1) = \beta\}$ and
$V_h^0 = \{v\colon [0,1] \to \mathbb{R}:\ v \text{ is cont. pw. linear on } T_h, v(0) = v(1) = 0\}$, where as before T_h is a uniform partition with mesh $h = \frac{1}{m+1}$. Observe that $V_h = \operatorname{span}(\varphi_0, \varphi_1, \dots, \varphi_m, \varphi_{m+1}) \subset V$ and $V_h^0 = \operatorname{span}(\varphi_1, \dots, \varphi_m) \subset V^0$ with the hat functions φ_j.
The FE problem then reads
$$\text{Find } u_h \in V_h \quad \text{such that} \quad \int_0^1 u_h'(x)\chi'(x)\,\mathrm{d}x + 4\int_0^1 u_h(x)\chi(x)\,\mathrm{d}x \quad \forall \chi \in V_h^0.$$
3. Choosing $\chi = \varphi_i$, writing $u_h(x) = \sum_{j=0}^{m+1} \zeta_j \varphi_j(x)$ with $\zeta_0 = \alpha$ and $\zeta_{m+1} = \beta$ (due to the non-homogeneous Dirichlet BC), and inserting everything into the FE problem gives the following linear system of equations
$$(S + 4M)\zeta = b,$$
where the $m \times m$ stiffness matrix S has entries $s_{ij} = \int_0^1 \varphi_i'(x)\varphi_j'(x)\,\mathrm{d}x$, see Chapter 3 for details, the $m \times m$ mass matrix M has entries $m_{ij} = \int_0^1 \varphi_i(x)\varphi_j(x)\,\mathrm{d}x$, and the $m \times 1$ vector b has entries $b_i = -\alpha(\varphi_0', \varphi_i')_{L^2} - \beta(\varphi_{m+1}', \varphi_i')_{L^2} - 4\alpha(\varphi_0, \varphi_i)_{L^2} - 4\beta(\varphi_{m+1}, \varphi_i')_{L^2}$. The entries of the matrices S and M as well as of the vector b can be computed exactly.
Solving this system gives the vector ζ and in turns the FE approximation u_h.

- Let us finally consider the problem of finding a numerical approximation of solutions to the BVP
$$\begin{cases} -au''(x) + bu'(x) = r & \text{for} \quad x \in (0,1) \\ u(0) = 0 \quad \text{and} \quad u'(1) = \beta, \end{cases}$$
where $\beta \neq 0$, $a, b > 0$, and r are given real number. One has a homogeneous Dirichlet boundary conditions for $x = 0$ and non-homogeneous Neumann boundary conditions for $x = 1$.

For ease of presentation we take $a = b = r = 1$ and derive a FE approximation as follows

1. Define the space $V = \{v\colon [0,1] \to \mathbb{R}:\ v, v' \in L^2(0,1), v(0) = 0\}$. Multiply the DE with a test function $v \in V$, integrate over the domain $[0,1]$ and get the VF
$$\text{Find } u \in V \quad \text{such that} \quad (u', v')_{L^2} + (u', v)_{L^2} = \int_0^1 v(x)\,\mathrm{d}x + \beta v(1) \quad \forall v \in V.$$
2. Next, define the finite dimensional space $V_h = \{v\colon [0,1] \to \mathbb{R}:\ v \text{ is cont. pw. linear on } T_h, v(0) = 0\}$, where as before T_h is a uniform partition with mesh $h = \frac{1}{m+1}$.
Observe that $V_h = \operatorname{span}(\varphi_1, \dots, \varphi_m, \varphi_{m+1}) \subset V$, with the hat functions φ_j.
The FE problem then reads
$$\text{Find } u_h \in V_h \quad \text{such that} \quad (u_h', \chi')_{L^2} + (u_h', \chi)_{L^2} = \int_0^1 \chi(x)\,\mathrm{d}x + \beta\chi(1) \quad \forall v \in V_h.$$

3. Choosing $\chi = \varphi_i$, writing $u_h(x) = \sum_{j=1}^{m+1} \zeta_j \varphi_j(x)$, observing that φ_{m+1} is a half hat function, and inserting everything into the FE problem gives the following linear system of equations

$$(S + C)\zeta = b,$$

where the $(m+1)\times(m+1)$ stiffness matrix S has entries $s_{ij} = \int_0^1 \varphi_i'(x)\varphi_j'(x)\,\mathrm{d}x$, the $(m+1)\times(m+1)$ convection matrix C has entries $c_{ij} = \int_0^1 \varphi_j'(x)\varphi_i(x)\,\mathrm{d}x$, and the $(m+1)\times 1$ vector b has entries $b_i = \int_0^1 \varphi_i(x)\,\mathrm{d}x + \beta\varphi_i(1)$. Detailed formulas for these entries can be found in the book (Section 5.3). Solving this system gives the vector ζ and in turns the FE approximation u_h.

- Let $f\colon (0,1) \to \mathbb{R}$ be bounded and continuous. Then, the BVP

$$\begin{cases} -u''(x) = f(x) & \text{for} \quad x \in (0,1) \\ u(0) = 0 \quad \text{and} & u(1) = 0 \end{cases}$$

  is equivalent to the VF

$$\text{Find} \quad u \in \mathscr{C}^2(0,1) \cap H_0^1 \quad \text{such that} \quad (u', v')_{L^2(0,1)} = (f, v)_{L^2(0,1)} \quad \text{for all} \quad v \in H_0^1.$$

- Poincaré inequality reads: Let $L > 0$ and consider the open interval $\Omega = (0, L)$. Assume that $u \in H_0^1(\Omega) = \{v\colon \Omega \to \mathbb{R}\colon\ v, v' \in L^2(\Omega), v(0) = v(L) = 0\}$. Then, one has

$$\|u\|_{L^2(\Omega)} \leq C_L \left\|u'\right\|_{L^2(\Omega)}.$$

- A priori error estimate in the energy norm. Let $f\colon (0,1) \to \mathbb{R}$ be bounded and continuous. Consider the BVP

$$\begin{cases} -u''(x) = f(x) & \text{for} \quad x \in (0,1) \\ u(0) = 0 \quad \text{and} & u(1) = 0. \end{cases}$$

  Denote by u_h the solution to the corresponding FE problem (cG(1) FE). Assume that $u \in \mathscr{C}^2(0,1)$. Then, there exists a $C > 0$ such that

$$\|u - u_h\|_E \leq Ch \left\|u''\right\|_{L^2(0,1)},$$

  where $\|v\|_E = \sqrt{(v,v)_E} = \sqrt{(v',v')_{L^2(0,1)}}$ denotes the energy norm.

- For indication, and for a uniform partition of $[0,1]$ denoted by T_h: $x_0 = 0 < x_1 < x_2 < \ldots < x_m < x_{m+1} = 1$ with element length/mesh denoted by h, we summarise the possible choices for the FE spaces:

  1. Dirichlet BC $u(0) = 0, u(1) = 0$: test and trial spaces given by $span(\varphi_1, \ldots, \varphi_m)$.
  2. Dirichlet BC $u(0) = \alpha \neq 0, u(1) = 0$: trial given by $span(\varphi_0, \varphi_1, \ldots, \varphi_m)$ and test by $span(\varphi_1, \ldots, \varphi_m)$.
  3. Dirichlet BC $u(0) = 0, u(1) = \beta \neq 0$: trial given by $span(\varphi_1, \ldots, \varphi_m, \varphi_{m+1})$ and test by $span(\varphi_1, \ldots, \varphi_m)$.
  4. Dirichlet BC $u(0) = \alpha \neq 0, u(1) = \beta \neq 0$: trial given by $span(\varphi_0, \varphi_1, \ldots, \varphi_{m+1})$ and test by $span(\varphi_1, \ldots, \varphi_m)$.
  5. Dirichlet/Neumann BC $u(0) = 0, u'(1) = \beta$ (zero or not): trial given by $span(\varphi_1, \ldots, \varphi_{m+1})$ and test by $span(\varphi_1, \ldots, \varphi_{m+1})$.

6. Neumann/Dirichlet BC $u'(0) = \alpha$ (zero or not), $u(1) = 0$: trial given by $span(\varphi_0, \dots, \varphi_m)$ and test by $span(\varphi_0, \dots, \varphi_m)$.
7. Dirichlet/Neumann BC $u(0) = \alpha \neq 0, u'(1) = \beta$ (zero or not): trial given by $span(\varphi_0, \dots, \varphi_{m+1})$ and test by $span(\varphi_1, \dots, \varphi_{m+1})$.
8. Neumann/Dirichlet BC $u'(0) = \alpha$ (zero or not), $u(1) = \beta \neq 0$: trial given by $span(\varphi_0, \dots, \varphi_{m+1})$ and test by $span(\varphi_0, \dots, \varphi_m)$.
9. Neumann BC $u'(0) = \alpha, u'(1) = \beta$ (zero or not): trial given by $span(\varphi_0, \dots, \varphi_{m+1})$ and test by $span(\varphi_0, \dots, \varphi_{m+1})$.

Further resources:

- www.simscale.com
- wiki
- wiki
- cs.uchicago.edu
- youtube