

MSG800/MVE170 Basic Stochastic Processes

Written re-exam Monday 11 April 2022 8.30-12.30

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AIDS: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

GRADES: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Show that if for a discrete time Markov chain state i is recurrent and does not communicate with state j in sense of Hsu book (i.e., $i \not\leftrightarrow j$), then $p_{ij} = 0$. **(5 points)**

Task 2. Let $\{X(t)\}_{t \geq 0}$ be a Wiener process with drift coefficient $\mu \in \mathbb{R}$ and $\mathbf{Var}\{X(1)\} = \sigma^2 > 0$. Find the conditional probability density function of $X(t)$ given that $X(s) = x$ for $0 < s < t$. **(5 points)**

Task 3. Let X_1, X_2, \dots be independent random variables with possible values $\{-1, 1\}$ and $\mathbf{P}\{X_i = -1\} = q = 1 - p$ and $\mathbf{P}\{X_i = 1\} = p$. Show that $\{Y_n\}_{n=0}^\infty$ given by $Y_n = (q/p)^{X_1 + \dots + X_n}$ for $n \geq 1$ and $Y_0 = 1$ is a martingale. **(5 points)**

Task 4. Consider a taxi station where taxis and customers arrive in accordance with independent Poisson processes with respective rates of one and two per minute. A taxi will wait no matter how many other taxis are present. However, an arriving customer that does not find a taxi waiting leaves. Find (a) the average number of taxis waiting, and (b) the proportion of arriving customers that get taxis. **(5 points)**

Task 5. The surface of a bacterium consists of several sites at which foreign molecules – some acceptable and some not – become attached. At a particular site molecules arrive according to a Poisson process with rate $\lambda > 0$. Among these molecules a proportion $\alpha \in (0, 1)$ is acceptable. Acceptable molecules stay at the site an exponentially distributed time with expected value $1/\mu_1 > 0$ while unacceptable molecules remain at the site for an exponential distributed time with expected value $1/\mu_2 > 0$. An arriving molecule will become attached only if the site is free of other molecules. Write up equations that determine what percentage of time the site is occupied with acceptable and unacceptable molecule, respectively. (The equations need not be solved.) **(5 points)**

Task 6. Show that a time discrete Markov chain that possesses a stationary distribution which it is started according to is stationary. **(5 points)**

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Task 1. If $p_{ij} > 0$, then $p_{ji}(n) = 0$ for all n as otherwise i and j would communicate. But then the process starting in i has a probability at least $p_{ij} > 0$ of never returning to i which contradicts the recurrence of i .

Task 2. We may write $X(t) = \sigma W(t) + \mu t$ where $W(t)$ is a standard Wiener process so that $(X(t)|X(s) = x) = (\sigma W(t) + \mu t | \sigma W(s) + \mu s = x) = \sigma (W(t) - W(s)) + \mu t + (\sigma W(s) | \sigma W(s) = x - \mu s)$ so that $(X(t)|X(s) = x)$ is $N(x + \mu(t-s), \sigma^2(t-s))$ -distributed as $W(t) - W(s)$ and $W(s)$ are independent.

Task 3. This is Task 5.105 in the Hsu book which is a solved exercise on the course web page.

Task 4. The number of taxis waiting $\{X(t)\}_{t \geq 0}$ is a M/M/1 queueing system with $\lambda = 1$ and $\mu = 2$. Therefore (a) $\mathbf{E}\{X(t)\} = \lambda/(\mu - \lambda) = 1/2$ and (b) the proportion of arrival of customers that find at least one taxi waiting $1 - p_0 = \lambda/\mu = 1/2$.

Task 5. Let the states be 0: no molecule attached, 1: an acceptable molecule is attached, and 2: an unacceptable molecule is attached. We have a birth and death process with generator

$$G = \begin{pmatrix} -\lambda & \alpha\lambda & (1-\alpha)\lambda \\ \mu_1 & -\mu_1 & 0 \\ \mu_2 & 0 & -\mu_2 \end{pmatrix}$$

and stationary distribution $\pi = (\pi_0 \ \pi_1 \ \pi_2)$ given by $\pi G = 0$ and $\pi_0 + \pi_1 + \pi_2 = 1$.

Task 6. Writing π for a stationary distribution and $\mu^{(n)}$ for the distribution at time n (so that $\mu^{(n)} = \pi$) we have $\mathbf{P}\{X_{n+k} = x_n, \dots, X_k = x_0\} = \mu_{x_0}^{(k)} p_{x_0 x_1} \cdot \dots \cdot p_{x_{n-1} x_n} = \pi_{x_0} p_{x_0 x_1} \cdot \dots \cdot p_{x_{n-1} x_n}$ which is independent of the time translation k .