MSG800/MVE170 Basic Stochastic Processes

Written re-exam Tuesday 23 August 2022 2-6 PM

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Aids: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

Grades: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

Motivations: All answers/solutions must be motivated. Good Luck!

Task 1. Consider a discrete time Markov chain $\{X(n)\}_{n=0}^{\infty}$ with state space $\{0,1,2,3\}$, starting value $X(0)=0$ and transition probabilities $p_{ij}=1/4$ for all i,j. Find the expected value of the time it takes the chain to reach the state 3. **(5 points)**

Task 2. Writing $\{W(t)\}_{t\geq 0}$ for a Wiener process with $\mathbf{E}\{W(1)^2\}=1$, is there a (non-random) function $f:[0,\infty)\to\mathbb{R}$ such that $\{\int_0^t W(r)^2\,dr+f(t)\}_{t\geq 0}$ is a martingale with respect to the history of the W-process $F_s=\sigma(W(r):r\in[0,s])$? **(5 points)**

Task 3. Find the variance of the random variable $X(t)+X'(t)$ when $X(t)$ is a zero-mean WSS process with autocorrelation function $R_X(\tau)=\mathrm{e}^{-\tau^2/2}$. **(5 points)**

Task 4. Let $\{X(t)\}_{t\geq 0}$ be a Poisson process with intensity $\lambda>0$ and write $T_1,T_2,\ldots$ for the independent exponentially distributed with parameter λ times between its jumps from 0 to 1, from 1 to 2, etc. Then $\sum_{i=1}^n T_i$ is a continuously gamma-distributed random variable with probability density function $f_{\sum_i^n T_i}(x)=\lambda^n x^{n-1}\mathrm{e}^{-\lambda x}/((n-1)!)$ for $x\geq 0$. Use this to prove that $X(t)$ is Poisson distributed with parameter λt. (Hint: $\mathbf{P}\{X(t)=n\}=\mathbf{P}\{\sum_{i=1}^n T_i\leq t<\sum_{i=1}^{n+1} T_i\}$.) **(5 points)**

Task 5. The input to a continuous time LTI system is a WSS continuous time random process $X(t)$ with autocorrelation function $R_X(\tau)=\frac{\sin(a\tau)}{\pi\tau}$ for $\tau\in\mathbb{R}$ for a constant $a>0$, while the LTI system has impulse response of the same form $h(t)=\frac{\sin(bt)}{\pi t}$ for $t\in\mathbb{R}$ for a constant $b>0$. Find the autocorrelation function of the output $Y(t)$ from the system. **(5 points)**

Task 6. Find the probability distribution of the time it takes a continuous time Markov chain $\{X(t)\}_{t\geq 0}$ with state space $\{0,1,2,3\}$ and generator $g_{ij}=1/3$ for $i\neq j$ to move from the starting value $X(0)=0$ to the value 3. **(5 points)**

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Task 1. Writing T for the time of consideration and Y for a waiting time distribution with parameter $p = 3/4$ so that $\mathbf{P}\{Y = k\} = (1/4)^{k-1}(3/4)$ for $k = 1, 2, \ldots$ we have $\mathbf{E}\{T\} = \mathbf{E}\{Y\} + (2/3) \cdot \mathbf{E}\{T\}$ giving $\mathbf{E}\{T\} = 3\,\mathbf{E}\{Y\} = 4$.

Task 2. $\mathbf{E}\{\int_0^t W(r)^2\,dr + f(t)|F_s\} = \int_0^s W(r)^2\,dr + \mathbf{E}\{\int_s^t (W(r) - W(s))^2\,dr + (t-s)W(s)^2 + 2W(s)\int_s^t (W(r) - W(s))\,dr|F_s\} + f(t) = \int_0^s W(r)^2\,dr + \int_s^t (r-s)\,dr - (t-s)W(s)^2 + 0 + f(t) \neq \int_0^s W(r)^2\,dr + f(s)$ for non-random functions f, so the answer is NO.

Task 3. According to exercises 6.7 and 6.8 in the Hsu book $\mu_{X'}(t) = \mu_X'(t) = 0$, $R_{X,X'}(\tau) = R_X'(\tau) = -\tau\,\mathrm{e}^{-\tau^2/2}$ and $R_{X'}(\tau) = -R_X''(\tau) = (1 - \tau^2)\,\mathrm{e}^{-\tau^2/2}$, so that $\mathbf{Var}\{X(t) + X'(t)\} = \mathbf{E}\{(X(t) + X'(t))^2\} = R_X(0) + 2\,R_{X,X'}(0) + R_{X'}(0) = 1 + 0 + 1$.

Task 4. $\mathbf{P}\{X(t) = 0\} = \mathbf{P}\{t < T_1\} = \mathrm{e}^{-\lambda t}$ and $\mathbf{P}\{X(t) = n\} = \mathbf{P}\{\sum_{i=1}^n T_i \leq t < \sum_{i=1}^{n+1} T_i\} = \int_0^t f_{\sum_i^n T_i}(x)\,\mathbf{P}\{T_{n+1} > t-x\}\,dx = \int_0^t \frac{\lambda^n x^{n-1}}{(n-1)!}\,\mathrm{e}^{-\lambda x}\,\mathrm{e}^{-\lambda(t-x)}\,dx = \frac{(\lambda t)^n}{n!}\,\mathrm{e}^{-\lambda t}$ for $n \geq 1$.

Task 5. We have $S_X(\omega) = 1$ for $|\omega| \leq a$ and $S_X(\omega) = 0$ otherwise while $H(\omega) = 1$ for $|\omega| \leq b$ and $H(\omega) = 0$ otherwise, so that $S_Y(\omega) = |H(\omega)|^2 S_X(\omega) = 1$ for $|\omega| \leq \min(a, b)$ and $S_Y(\omega) = 0$ otherwise which corresponds to $R_Y(\tau) = \frac{\sin(\min(a,b)\tau)}{\pi\tau}$ for $\tau \in \mathbb{R}$.

Task 6. Writing T for the time under consideration and using that $X(t)$ spends a unit parameter exponentially distributed time at each state we get $\Psi_T(\omega) = \Psi_{\exp(1)}(\omega) \cdot ((1/3) \cdot 1 + (2/3) \cdot \Psi_T(\omega))$ so that $\Psi_T(\omega) = \frac{(1/3)\cdot\Psi_{\exp(1)}(\omega}{1-(2/3)\cdot\Psi_{\exp(1)}(\omega)} = \frac{1/3}{1/3 - j\omega}$ making T exponentially distributed with parameter $1/3$.