

MVE172 Basic Stochastic Processes and Financial Applications

Written re-exam Tuesday 23 August 2022 2-5 PM

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AIDS: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

GRADES: 8, 12 and 16 points for grades 3, 4 and 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Consider a discrete time Markov chain $\{X(n)\}_{n=0}^{\infty}$ with state space $\{0, 1, 2, 3\}$, starting value $X(0) = 0$ and transition probabilities $p_{ij} = 1/4$ for all i, j . Find the expected value of the time it takes the chain to reach the state 3. **(5 points)**

Task 2. Writing $\{W(t)\}_{t \geq 0}$ for a Wiener process with $\mathbf{E}\{W(1)^2\} = 1$, is there a (non-random) function $f : [0, \infty) \rightarrow \mathbb{R}$ such that $\{\int_0^t W(r)^2 dr + f(t)\}_{t \geq 0}$ is a martingale with respect to the history of the W -process $F_s = \sigma(W(r) : r \in [0, s])$? **(5 points)**

Task 3. Find the variance of the random variable $X(t) + X'(t)$ when $X(t)$ is a zero-mean WSS process with autocorrelation function $R_X(\tau) = e^{-\tau^2/2}$. **(5 points)**

Task 4. Let $\{X(t)\}_{t \geq 0}$ be a Poisson process with intensity $\lambda > 0$ and write $T_1, T_2, \dots$ for the independent exponentially distributed with parameter λ times between its jumps from 0 to 1, from 1 to 2, etc. Then $\sum_{i=1}^n T_i$ is a continuously gamma-distributed random variable with probability density function $f_{\sum_{i=1}^n T_i}(x) = \lambda^n x^{n-1} e^{-\lambda x} / ((n-1)!) for $x \geq 0$. Use this to prove that $X(t)$ is Poisson distributed with parameter λt . (HINT: $\mathbf{P}\{X(t) = n\} = \mathbf{P}\{\sum_{i=1}^n T_i \leq t < \sum_{i=1}^{n+1} T_i\}$.) **(5 points)**$

MVE172 Solutions to written re-exam 23 August 2022

Task 1. Writing T for the time of consideration and Y for a waiting time distribution with parameter $p = 3/4$ so that $\mathbf{P}\{Y = k\} = (1/4)^{k-1}(3/4)$ for $k = 1, 2, \dots$ we have $\mathbf{E}\{T\} = \mathbf{E}\{Y\} + (2/3) \cdot \mathbf{E}\{T\}$ giving $\mathbf{E}\{T\} = 3\mathbf{E}\{Y\} = 4$.

Task 2. $\mathbf{E}\{\int_0^t W(r)^2 dr + f(t) | F_s\} = \int_0^s W(r)^2 dr + \mathbf{E}\{\int_s^t (W(r) - W(s))^2 dr + (t-s)W(s)^2 + 2W(s) \int_s^t (W(r) - W(s)) dr | F_s\} + f(t) = \int_0^s W(r)^2 dr + \int_s^t (r-s) dr - (t-s)W(s)^2 + 0 + f(t) \neq \int_0^s W(r)^2 dr + f(s)$ for non-random functions f , so the answer is NO.

Task 3. According to exercises 6.7 and 6.8 in the Hsu book $\mu_{X'}(t) = \mu'_X(t) = 0$, $R_{X,X'}(\tau) = R'_X(\tau) = -\tau e^{-\tau^2/2}$ and $R_{X'}(\tau) = -R''_X(\tau) = (1 - \tau^2)e^{-\tau^2/2}$, so that $\mathbf{Var}\{X(t) + X'(t)\} = \mathbf{E}\{(X(t) + X'(t))^2\} = R_X(0) + 2R_{X,X'}(0) + R_{X'}(0) = 1 + 0 + 1$.

Task 4. $\mathbf{P}\{X(t) = 0\} = \mathbf{P}\{t < T_1\} = e^{-\lambda t}$ and $\mathbf{P}\{X(t) = n\} = \mathbf{P}\{\sum_{i=1}^n T_i \leq t < \sum_{i=1}^{n+1} T_i\} = \int_0^t f_{\sum_{i=1}^n T_i}(x) \mathbf{P}\{T_{n+1} > t-x\} dx = \int_0^t \frac{\lambda^n x^{n-1}}{(n-1)!} e^{-\lambda x} e^{-\lambda(t-x)} dx = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$ for $n \geq 1$.