MATHEMATICIZATION!

1.

MATH AND MUSIC

Can math explain why a guitar sounds the way it does?

VIBRATIONS CREATE SOUND

- The strings on a guitar vibrate.
- The vibration makes a soundwave.
- The soundwave pushes the air molecules and eventually reaches our ears.

THE SOUND DEPENDS ON…

- The length of the string.
- How it was set into motion (plucked, strummed, tapped, etc).
- The shape and composition of the string.
- We can *mathematicize* a vibrating guitar string.

Mathematicization of a Vibrating String

$u_{tt}(x,t) = u_{xx}(x,t)$ (the wave equation)

$u(0,t) = u(L, t) = 0$ (the ends don't move)

$u(x,0) = f(x)$ (how the string is set in motion)

$u_t(x,0) = 0$ (not moving at $t=0$)

MULTIVARIABLE CALCULUS???

- The function $u(x,t)$ is the height of the string at time $= t$, position $= x$.
- u depends on *two independent variables.*

MULTIVARIABLE -> SINGLE VARIABLE ???

- How could we turn this into problem(s) for function(s) that depend on one variable only?
- Discuss!

Separation of Variables

- See if we can somehow build a solution from one-dimensional problems…
- $T''(t)\,X(x) = X''(x)\,T(t)$ … re-arrange …
- $T(t)\,X(0) = 0$, $T(t)\,X(L) = 0$ means ?

WE NOW LOOK FOR…

- functions that satisfy $X''(x) = c\,X(x)$, where c can be any real number.
- also need $X(0) = X(L) = 0$.
- can you find some real numbers c and functions X?

THE SPECTRUM

Solutions are

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right) \mathit{with} -\lambda_n = n^2\pi^2/L^2$$

The numbers $\lambda_n = n^2\pi^2/L^2$ are eigenvalues.

The set of all eigenvalues is *the spectrum.*

FREQUENCIES

The spectrum determines the frequencies of vibration of the string.

$$\frac{\pi^2}{L^2}, \frac{4\pi^2}{L^2}, \frac{9\pi^2}{L^2}, \frac{16\pi^2}{L^2}, \ldots\ldots$$

If L is shorter, these numbers get bigger….

Hearing the Length of a String

Shorter strings make higher notes.

Longer strings make lower notes.

We can see this in the guitar solo...

How can we be sure these are all?

- Fourier 1768–1830
- Could there be more?
- How to get $u(x,t)$???

Geometric representation of functions

- 1862–1943
- infinite dimensional vector space with scalar product
- Hilbert space

ORTHOGONAL BASIS FOR HILBERT SPACE

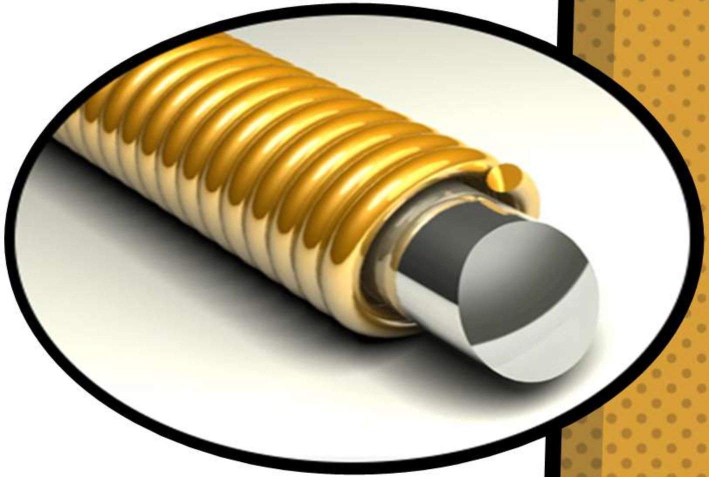
- The functions $X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ are an OB for a Hilbert space that contains all solns.
- Expand any function in that space using them…

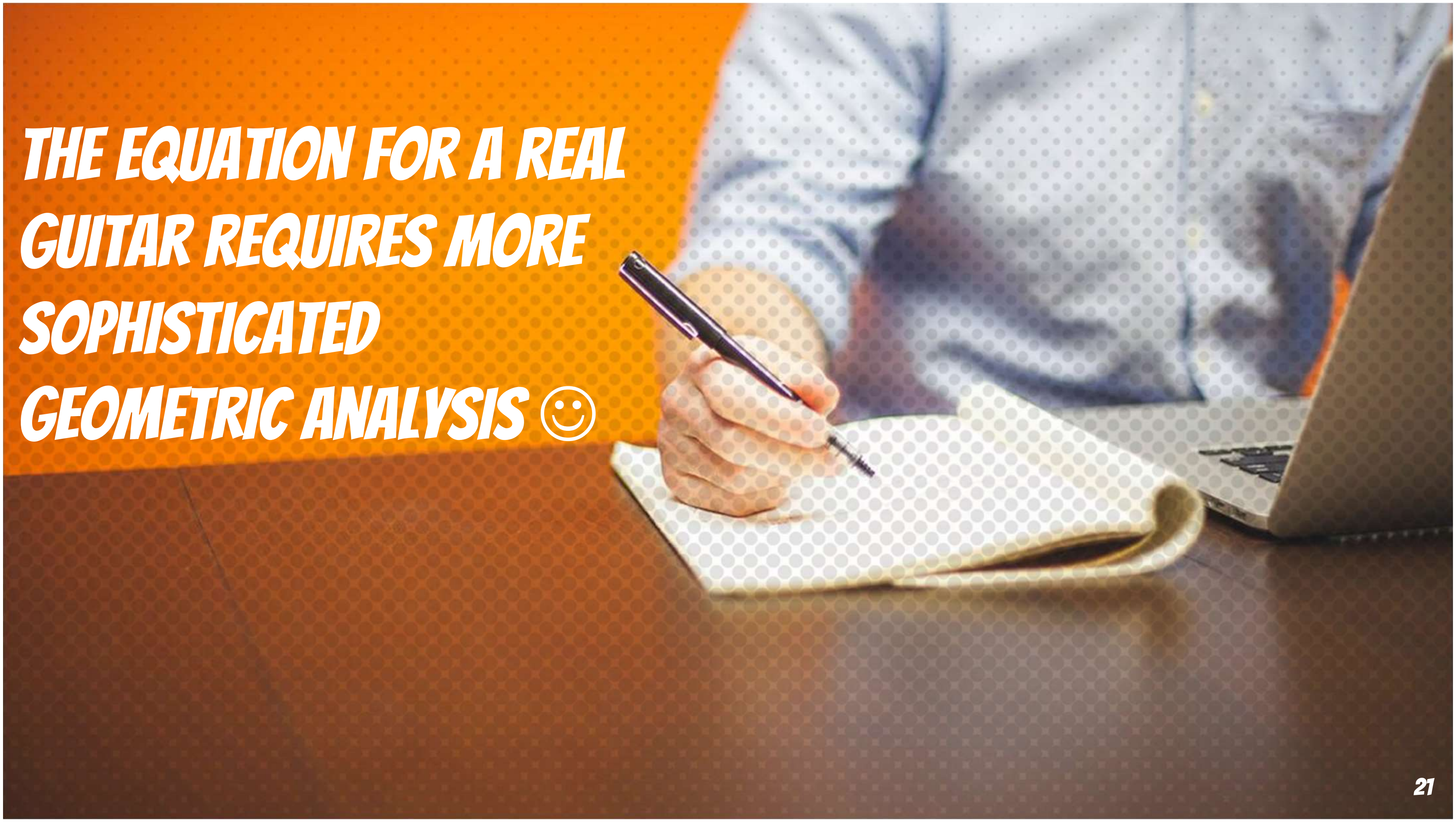
Solution to the vibrating string

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi t}{L}\right) f_n$$

$$f_n = \frac{4}{L^2} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

A Real Guitar String is More Complicated




THE EQUATION FOR A REAL GUITAR REQUIRES MORE SOPHISTICATED GEOMETRIC ANALYSIS ☺
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GEOMETRIC ANALYSIS

PHYSICS + GEOMETRY REQUIRES GA.

GA IS A WAY TO APPROACH ALL PROBLEMS...

MATHEMATICIZATION + GEOMETRIZATION...

GAME THEORY

Mathematics that can explain why people and animals act they way they do!

ROCK PAPER SCISSORS!

MATHEMATICIZATION!

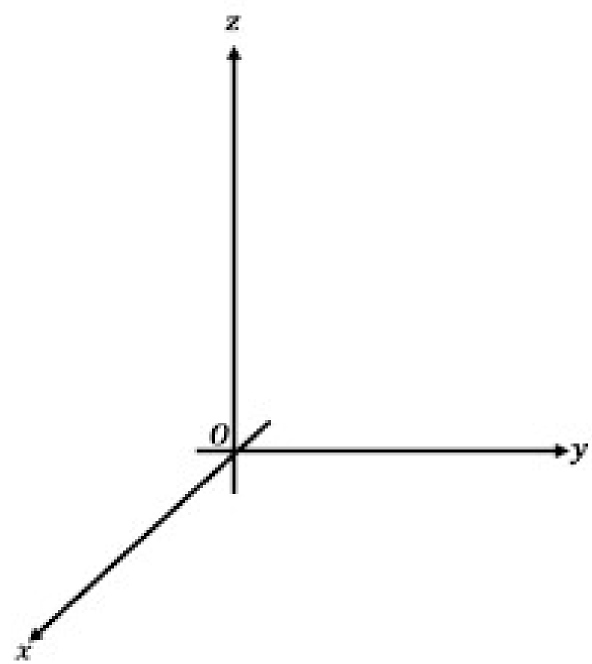
	Player 1 Rock	Player 1 Paper	Player 1 Scissors
Player 2 Rock	(0, 0)	(-1, 1)	(1, -1)
Player 2 Paper	(1, -1)	(0, 0)	(-1, 1)
Player 2 Scissors	(-1, 1)	(1, -1)	(0, 0)

Pure and Mixed Strategies

- Pure strategies are doing one thing all the time.
- Mixed strategies are doing each of the pure strategies with a certain frequency.
- Payoff depends on all (mixed) strategies.

GEOMETRIZATION

- (r, p, s) probabilities for player 1.
- (ρ, π, σ)probabilities for player 2.
- calculate payoffs!

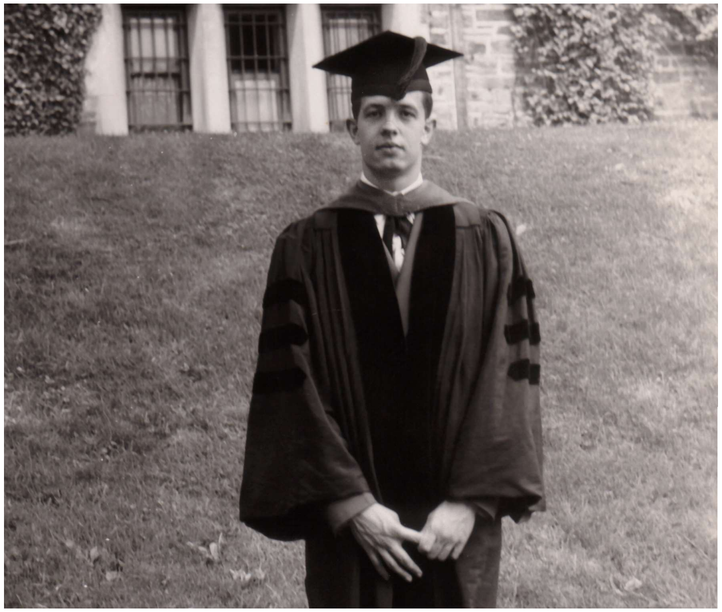


EQUILIBRIUM STRATEGY

No player can increase their payoff by changing their strategy, while other players keep their strategies fixed.

EQUILIBRIUM STRATEGIES EXIST!

- JF Nash, 1950
- Nobel prize 1994
- A beautiful mind
- Abel prize 2015
- Embedding theorem




Acceptance speech by John Nash

EQUILIBRIUM STRATEGY?

PRISONER'S DILEMMA

Two prisoners are caught for a crime.

They agreed to remain silent.

Will they keep their promise or rat each other out?

Discuss!

Mathematicization

R=reward payoff for cooperating

P=punishment payoff for mutual defection

T=temptation payoff

S=sucker's payoff

$S<P<R<T$

Strategy	Cooperate	Defect
Cooperate	(R, R)	(S, T)
Defect	(T, S)	(P, P)

EQUILIBRIUM STRATEGY?

	Cooperate	Defect
Cooperate	(0, 0)	(-10, -2)
Defect	(-2, -10)	(-5, -5)

APPLICATION: THE CLIMATE DILEMMA

- If China reduces emissions but USA doesn't, USA profits (T), China loses (S).
- If both don't reduce emissions, both (P).
- If both reduce emissions both (R).
- S<P<R<T

Mathematicization

Strategy	Cooperate	Defect
Cooperate	(R, R)	(S, T)
Defect	(T, S)	(P, P)

EVOLUTION OF BEHAVIORS

- x is frequency of cooperating countries
- 1-x is frequency of defecting
- replicator equation predicts evolution of behavior (cooperation or not)

REPLICATOR EQUATION

$$\dot{x} = x\begin{bmatrix} xR + (1-x)S \ - x(xR + (1-x)S) \\ -(1-x)(xT + (1-x)P) \end{bmatrix}$$

$$= -x(1-x)[x(T-R) + (1-x)(P-S)] \leq 0.$$


WHAT ABOUT US?

THERE IS SOME HOPE… THE OPTIMAL REPLICATOR EQUATION

Evolution with two mechanisms

- countries choose selfishly at a fast time scale
- but at a long time scale, global populations with the higher final average fitness are favored

Problem in optimal control theory

- https://www.nature.com/articles/s41598-018-20426-w

$$2P \leq T \leq 2R$$

if this eqn holds, evolution towards cooperation.

APPLICATIONS!

BIOLOGY
animal behavior, biodiversity, ecosystems

ECONOMICS
decision making, business strategies

SOCIOLOGY
human behaviors and consequences


TACK! ANY QUESTIONS?