Lecture_3

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Options and Mathematics: Lecture 3

November 5, 2020

I BUY THE ZCB AT $t=0$ FOR 0.9 \$ ← LEND 0.9 \$ TO THE SELLER AT $t=0$

AT MATURITY T I RECEIVE 1 \$

Basic financial concepts

Zero Coupon Bonds

The **zero-coupon bond** (ZCB) with **face** (or **nominal**) **value** K and maturity $T > 0$ is the contract that promises to pay to its owner the amount K at time T in the future.

Hence a ZCB is a European style derivative with pay-off $Y = K$.

Without loss of generality we assume from now on that $K = 1$, as owning one share of the ZCB with face value K is clearly equivalent to own K shares of the ZCB with face value 1

ZCB's (and the related coupon bonds described below) are first issued in the so-called **primary market** by national governments and private companies as a way to borrow money and fund their activities; starting from the following market day, the ZCB's become tradable assets in the **secondary** market and thus their price changes in time.

$S(t) \equiv$ STOCK PRICE

$B(t,T) \equiv$ ZCB PRICE

Let $B(t, T)$ denote the value at time t of the ZCB with face value 1 and expiring at time T. $t \leq T$

If the issuer of the ZCB announces at time $t_0 < T$ that it is unable to comply with the payment of the face value at maturity, then the ZCB becomes worthless, i.e., $B(t, T) \equiv 0$ for $t \in [t_0, T]$ and the issuer of the ZCB is said to be in **default**.

Suppose that the issuer of the ZCB bears no risk of default in the interval $[t, T]$. The investors who own shares of the ZCB at maturity T will then receive at time T the promised face value, multiplied by the number of shares owned, from the original issuer of the ZCB.

Suppose the ZCB is bought at time t and kept until maturity. The return per share of this investment is

$$R(t) = 1 - B(t, T).$$

IN THE INTERVAL $[t,T]$

Under normal market conditions, $B(t, T) < 1$, for $t < T$, i.e., we pay less than 1 today to receive 1 in the future, and thus $R(t) > 0$. However exceptions are possible; for instance national bonds in Sweden with maturity shorter than 5 years yield currently (2017) a negative return.

Coupon Bonds

Bonds with long maturity typically pay coupons in addition to the face value.

Let $0 < t_1 < t_2 < \cdots < t_M = T$ be a partition of the interval $[0, T]$.

A **coupon bond** with maturity T, face value 1 and coupons $c_1, c_2, \dots, c_M \in [0, 1)$ is a contract that promises to pay the amount c_k at time t_k and the amount $1 + c_M$ at maturity $T = t_M$.

REMARK: TYPICALLY COUPONS ARE ALL EQUALS $(c_1 = c_2 = \dots = c_M)$ AND PAID ANNUALLY

$B(t) \equiv$ VALUE AT TIME t OF A GENERIC MONEY MARKET ASSET

Money market

The **money market** is a component of the debt market consisting of **short term loans**, i.e., loan contracts with maturity between one day and one year.

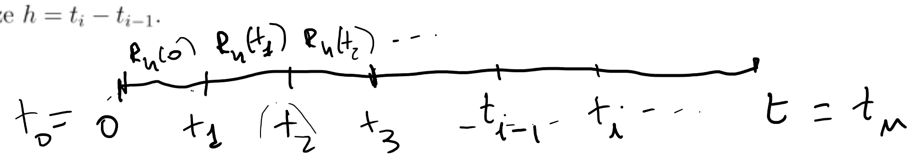
Examples of money market assets are **treasury bills**, i.e., ZCB's with maturity $T < 1$ year.

Other examples of money market assets are commercial papers, certificates of deposit, saving accounts and repurchase agreements (**repo**).

The value at time t of a generic asset in the money market will be denoted by $B(t)$.

The difference $B(t_2) - B(t_1)$, $t_1 < t_2$, determines the **interest rate** of the asset in the interval $[t_1, t_2]$.

Let $\{t_0 = 0, t_1, \dots, t_N = t\}$ be a uniform partition of the interval $[0, t]$ with size $h = t_i - t_{i-1}$.



The money market asset is said to have **simply compounded** interest rate $R_h(s)$ in the time period $[s, s+h]$, if

$$B(s+h) = B(s)(1 + R_h(s)h), \quad s \in \{t_0, \dots, t_{N-1}\}.$$

$B(t_1) = B(t_0)(1 + R_h(0)h)$

$B(t_2) = B(t_1)(1 + R_h(t_1)h)$

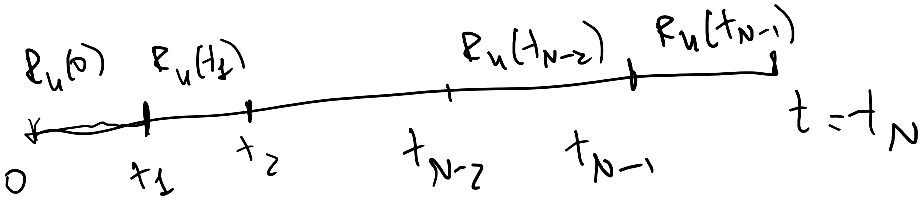
$= B(t_0)(1 + R_h(0)h) \times (1 + R_h(t_1)h)$

Inverting this equation we have

$$\boxed{R_h(s) = \frac{B(s+h) - B(s)}{hB(s)}},$$

SIMPLY COMPOUNDED IN THE INTERVAL $[s, s+h]$

Important: $R_h(s)$ *is known at time* s (as opposed for instance to the return of stocks in the interval $[s, s+h]$, which is not known at time s).



We can express the value at time $t = t_N$ of the risk-free asset in terms of the value at time $t = 0$ by the formula

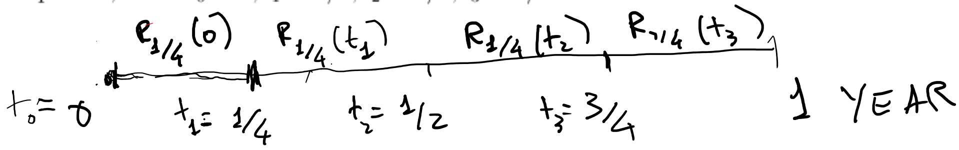
$$B(t) = B(t_{N-1})(1 + R_h(t_{N-1})h) = B(t_{N-2})(1 + R_h(t_{N-2})h)(1 + R_h(t_{N-1})h) = \cdots$$

$$= \cdots = B(0)\prod_{i=0}^{N-1}(1 + R_h(t_i)h).$$

Example.

Suppose that at time $t_0 = 0$ an investor is borrowing the quantity $B(0) = 1000000$ Kr for one year with 3-months compounded interest rate, i.e., $h = 1/4$.

Suppose $R_{1/4}(t_0) = 0.03$ in the first quarter, $R_{1/4}(t_1) = 0.02$ in the second quarter, $R_{1/4}(t_2) = 0.01$ in the third quarter and $R_{1/4}(t_3) = 0.04$ in the last quarter, where$t_0 = 0, t_1 = 1/4, t_2 = 1/2, t_3 = 3/4$.



The debt of the investor at time $t_4 = 1$ year is

DEBT AFTER 3 MONTHS

$$B(t_4) = B(t_0)(1 + \frac{1}{4}R_{1/4}(t_0))(1 + \frac{1}{4}R_{1/4}(t_1))(1 + \frac{1}{4}R_{1/4}(t_2))(1 + \frac{1}{4}R_{1/4}(t_3))$$

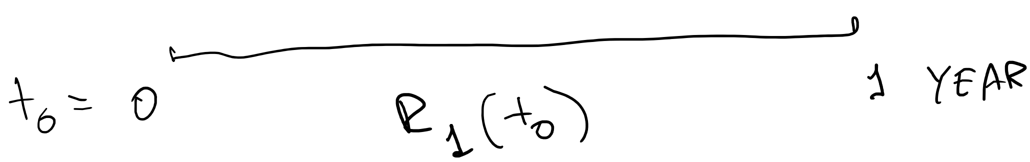
$$\approx 1025220 \text{ Kr}.$$

DEBT AFTER 6 MONTHS

If the investor borrows instead at the yearly compounded rate $R_1(t_0) = 0.03$ (i.e., $h = 1$), the debt after 1 year is

h=1 YEAR

$$B(t_4) = B(t_0)(1 + R_1(t_0)) = 1030000\ Kr.$$



Letting $h \to 0$ in the definition of $R_h(s)$ we obtain the **continuously compounded** interest rate (or **short rate**) $r(t)$ of the money market asset, namely

$$R_h(s) = \frac{B(s+h) - B(s)}{hB(s)} \to r(s) = \frac{B'(s)}{B(s)}, \quad \text{as } h \to 0$$

that is

$$\boxed{r(s) = \frac{d}{ds} \log B(s)},$$

$\int_0^t r(s)\, ds = \log B(t) - \log B(0)$

$r(t)$ is the interest rate to borrow at time t for an "infinitesimal" interval of time, which in the real world corresponds to overnight loans.

Integrating on the time interval $[0, t]$ we obtain

$$\boxed{B(t) = B(0) \exp\left(\int_0^t r(s)\, ds \right)},$$

TAKE THE EXPONENTIAL OF BOTH SIDES

Remark: If the short rate is constant, i.e., $r(t) \equiv r$, then $B(t) = B(0)e^{rt}$

Frictionless markets

We are going to introduce some simplifications, among which

1. There is no bid/ask spread
2. There are no transaction costs and trades occur instantaneously
3. An investor can trade any fraction of shares
4. When a stock pays a dividend, the ex-dividend date and the payment date are the same and the stock price at this date drops by the exact same amount paid by the dividend

An ideal market satisfying these properties is also callled **frictionless market**.

Remark: By Assumption 1, any offer to buy/sell an asset is matched by an offer to sell/buy the asset (provided of course the price is *fair*).

Remark: Thanks to assumption 3, perfect self-financing portfolio processes in frictionless markets always exist.

$C(t, S(t), K, T) \equiv$ PRICE AT TIME t OF EUROPEAN CALL WITH STRIKE K AND MATURITY T

$P(t, S(t), K, T) \equiv$ —— " —— EUROPEAN PUT...

$\hat{C}(t, S(t), K, T) \equiv$ —— " —— AMERICAN CALL ...

$\hat{P}(t, S(t), K, T) \equiv$ —— " — AMERICAN PUT ...

Basic principles of options pricing theory

1. no-dummy investor principle (or **rational investor principle**):

$\boxed{\text{Investors prefer more to less and do not undertake trading strategies which result in a sure loss.}}$

For instance, applying this principle we derive the following properties of options prices:

(i) The price of a financial derivative tends to its pay-off as maturity is approached. In particular, for European call/put options,

$$C(t, S(t), K, T) \to (S(T) - K)_+, \quad P(t, S(t), K, T) \to (K - S(T))_+,$$

$t \le T$

as $t \to T^-$ and similarly for American options, while for the ZCB with maturity T and face value 1 there holds

$$\lim_{t \to T^-} B(t, T) = 1.$$

(ii) An American derivative is at least as valuable as its European counterpart. In particular, for call/put options,

$$\widehat{C}(t, S(t), K, T) \ge C(t, S(t), K, T), \quad \widehat{P}(t, S(t), K, T) \ge P(t, S(t), K, T).$$

(iii) The price of an American derivative is always larger or equal to its intrinsic value. In particular, for American call/put options,

$t \le T$

$$\widehat{C}(t, S(t), K, T) \ge (S(t) - K)_+, \quad \widehat{P}(t, S(t), K, T) \ge (K - S(t))_+.$$

(iv) The price of European and American call options at time t is no larger than the price of the underlying asset at time t, i.e.,

$$\underline{C(t, S(t), K, T) \le S(t)} \quad \text{and} \quad \underline{\widehat{C}(t, S(t), K, T)} \le S(t).$$

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By (i) and integrating the equation

$$r(s) = \frac{d \log B(s)}{ds}$$

in the time interval $[t, T]$, we can write the value at time t of the ZCB as

$$\boxed{B(t,T) = e^{-\int_t^T r(s)\,ds}}.$$

2. Existence of risk-free assets

An asset $\mathcal{U}$ is said to be **risk-free** in the interval $[t, T]$ if the value of $\mathcal{U}$ at time T is known at time t.

Examples

- a ZCB with face value 1 and maturity $T > 0$ issued at time $t = 0$ satisfies $B(T, T) = 1$ and thus it is risk-free in any interval $[t, T]$.
- the rate of return of money market assets in a sufficiently short period of time $[t, t+h]$ is known at time t and so these assets are risk-free in the interval $[t, t+h]$.

However *ZCB's and money market assets can be considered risk-free only if the borrower party bears no risk of default.*

$$\boxed{\textit{There exist risk-free assets in the money market.}}$$

In a frictionless market the interest rate of all risk-free assets in the money market must necessarily be the same, otherwise one would generate a profit by borrowing at the lower rate and lending at the higher rate (this is an example of arbitrage opportunity, see definition below).

The (hypothetical) common short rate of all risk-free assets in the money market will be referred to as the **risk-free rate**.

$r(t)$

$= \frac{d}{dt} \log B(t)$

3. Arbitrage-free principle

Before introducing our next principle we need to discuss the fundamental concept of arbitrage portfolio process.

Definition

Let t be the present time and $T > t$. A portfolio process $\mathcal{A}$ is called an **arbitrage** in the interval $[t, T]$ if

(a) $V_{\mathcal{A}}(t) = 0$;

(b) It is known at time t that the return of $\mathcal{A}$ is positive in the interval $[t, T]$.

Hence an arbitrage portfolio is an investment strategy that requires no initial wealth and which ensures a positive profit without taking any risk.

Despite appearing "too good to be true", arbitrage opportunities do actually exist in real markets, but only for a very short time, as they are quickly exploited and "traded away" by investors (**arbitragers**).

In this course we neglect arbitrage opportunities altogether by imposing the **arbitrage-free principle:**

> *Asset prices are such that no arbitrage can be found in the market.*