Lecture_4

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Options and Mathematics: Lecture 4

November 6, 2020

Qualitative properties of option prices

Theorem 1.1 Assume that the arbitrage-free principle holds and let $\mathcal{A}$ be a portfolio process in the interval $[t, T]$.

Suppose that it is known at time t that the portfolio will generate the total cash flow $C_{\mathcal{A}}$ in the interval (t, T).

(a) If it is known at time t that $V_{\mathcal{A}}(T) \geq -C_{\mathcal{A}}$, then $V_{\mathcal{A}}(t) \geq 0$.

(b) If it is known at time t that $V_{\mathcal{A}}(T) = -C_{\mathcal{A}}$, then $V_{\mathcal{A}}(t) = 0$.

Proof. (a) Recall that the return of the portfolio in the interval $[t, T]$ is given by

$$R_{\mathcal{A}} = V_{\mathcal{A}}(T) - V_{\mathcal{A}}(t) + C_{\mathcal{A}}.$$

If it is known at time t that $V_{\mathcal{A}}(T) \geq -C_{\mathcal{A}}$, then it is known at time t that $R_{\mathcal{A}} \geq -V_{\mathcal{A}}(t)$.

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Assume (by contradiction) that $V_{\mathcal{A}}(t) < 0$. The latter means that after opening the portfolio process $\mathcal{A}$ at time t the investor is left with the cash $-V_{\mathcal{A}}(t)$.

The investor can then use this cash to add to the portfolio $\mathcal{A}$ at time t the number h of shares of a risk-free asset such that $hB(t) = -V_{\mathcal{A}}(t)$.

Let us call $\mathcal{A}'$ this new portfolio process. Then $\mathcal{A}'$ is an arbitrage, because its value at time t is zero and moreover at time t it is known that the return of $\mathcal{A}'$ in the interval $[t, T]$ satisfies

$$\begin{aligned} R_{\mathcal{A}'} &= R_{\mathcal{A}} + hB(T) - hB(t) = R_{\mathcal{A}} + hB(T) + V_{\mathcal{A}}(t) \\ &= V_{\mathcal{A}}(T) + C_{\mathcal{A}} + hB(T) \geq hB(T) > 0. \end{aligned}$$

Hence in a arbitrage-free market $V_{\mathcal{A}}(t) \geq 0$ must hold.

(b) We apply the result (a) to $-\mathcal{A}$, i.e.,

$$V_{-\mathcal{A}}(T) \geq -C_{-\mathcal{A}} \text{ implies } V_{-\mathcal{A}}(t) \geq 0.$$

As

$$C_{-\mathcal{A}} = -C_{\mathcal{A}} \text{ and } V_{-\mathcal{A}}(t) = -V_{\mathcal{A}}(t),$$

we obtain that

$$V_{\mathcal{A}}(T) \leq -C_{\mathcal{A}} \text{ implies } V_{\mathcal{A}}(t) \leq 0.$$

Combining the latter result with (a) completes the proof of (b). $\square$

Theorem 1.2

Assume that the arbitrage-free principle holds and that the underlying stock does not pay dividends in the interval (t, T).

$V_A(t) = 0$

(v) The **put-call parity** holds

$$\boxed{C(t, S(t), K, T) - P(t, S(t), K, T) = S(t) - KB(t, T)},$$

where $B(t, T)$ is the value at time t of the ZCB with face value 1 and maturity T.

CONVEX

$R(t) = 1 - B(t, T)$

(vi) If $B(t, T) \leq 1$, i.e., if the ZCB ensures a non-negative return, then $C(t, S(t), K, T) \geq (S(t) - K)_+$.

CONCAVE

(vii) If $B(t, T) \leq 1$, the map $T \to C(t, S(t), K, T)$ is non-decreasing.

(viii) If $K_0 \leq K_1$, then $C(t, S(t), K_0, T) \geq C(t, S(t), K_1, T)$, i.e., the price of European call options is non-increasing with the strike price. Similarly the price of put options is non-decreasing with the strike price.

(ix) The maps $K \to C(t, S(t), K, T)$ and $K \to P(t, S(t), K, T)$ are convex.

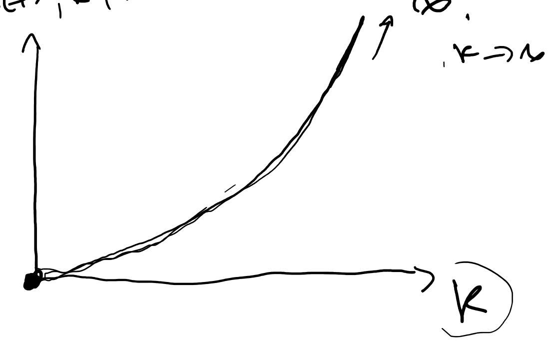
Remark: Recall that a real-valued function f on an interval I is convex if $f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta) f(y)$, for all $x, y \in I$ and $\theta \in (0, 1)$

FIXED

$C(t, S(t), K, T)$

PAY-OFF $(K - S(t))_+$

$P(t, S(t), K, T)$



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$\lim_{K \to \infty} C(t, S(t), K, T) = 0$

$\lim_{K \to \infty} P(t, S(t), K, T) = \infty$

$$C - P = S - KB \iff S - C + P - KB = 0$$

$A =$ 1 SHARE OF STOCK, -1 SHARE OF THE CALL, 1 SHARE OF THE PUT, $-K$ SHARES OF ZCB

$$V_A(t) = S(t) - C(t,S(t),K,T) + P(t,S(t),K,T) - KB(t,T)$$

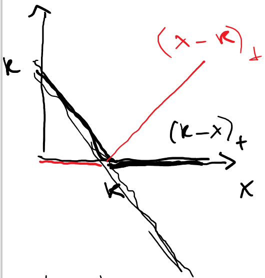
Proof of the put call parity. (v) Consider a constant portfolio $\mathcal{A}$ which is long one share of the stock and one share of the put option, and short one share of the call and K shares of the risk-free ZCB. The value of this portfolio at maturity is

REPLACE $t = T$ HERE

$$V_{\mathcal{A}}(T) = S(T) + (K - S(T))_+ - (S(T) - K)_+ - K = 0,$$

THE PUT-CALL PARITY IS EQUIVALENT TO $V_A(t) = 0$

where we used that $= \cancel{S(T)} + \cancel{K} - \cancel{S(T)} - \cancel{K} = 0$



$$\boxed{(K - x)_+ - (x - K)_+ = K - x \text{ for all } x \in \mathbb{R}}.$$

$(K-x)_+ - (x-K)_+ = K - x$

Since the portfolio $\mathcal{A}$ is constant and the stock does not pay dividends, then $\mathcal{A}$ is self-financing.

Using Theorem 1.1(b) with $C_{\mathcal{A}} = 0$ we conclude that $V_{\mathcal{A}}(t) = 0$, for $t < T$, that is

$$S(t) + P(t, S(t), K, T) - C(t, S(t), K, T) - KB(t, T) = 0,$$

which is the claim. □

Remark: In most of the course we assume that $r(t) \equiv$ is constant. In this case the value of the ZCB becomes

$$B(t, T) = e^{-r(T-t)}$$

and the put call parity reads

$$C(t, S(t), K, T) - P(t, S(t), K, T) = S(t) - Ke^{-r(T-t)}, \quad t \le T,$$

$t=0$ (today) $\quad t_* \quad T$

$\hat{P}(t, S(t), K, T)$

Optimal exercise of American put options

Consider a no-dummy investor owning an American put option.

Suppose that the investor wants to close the position on the American put at time t. This can be done by either selling the option or by exercising it.

In which case should the investor exercise the option? At any time $t < T$ we have, by (iii),

$$\text{either } \widehat{P}(t, S(t), K, T) > (K - S(t))_+ \quad \text{or} \quad \widehat{P}(t, S(t), K, T) = (K - S(t))_+.$$

Exercising the American put at a time t when the strict inequality holds is a dummy decision, because the income generated by exercising the option is lower than the amount that the buyer would receive by selling the option.

On the other hand, if the equality $\widehat{P}(t, S(t), K, T) = (K - S(t))_+$ holds at time t, then the exercise of the American put is optimal, as in this case the pay-off equals the value of the derivative, i.e., the investor takes full advantage of the American put.

This leads us to introduce the following definition.

Definition 1.2

A time $t < T$ is called an **optimal exercise time** for the American put with value $\widehat{P}(t, S(t), K, T)$ if $S(t) < K$ (i.e., the put is in the money) and

$$\widehat{P}(t, S(t), K, T) = (K - S(t))_+.$$

$$C - P = S - K\underbrace{e^{-r(T-t)}}_{B(t,T)} \Rightarrow C = P + S - Ke^{-r(T-t)} \geq S - Ke^{-r(T-t)}$$

WHEN THE RISK-FREE RATE IS CONSTANT

$$\widehat{C}(t, S(t), K, T) \geq (S(t) - K)_+$$

Optimal exercise of American call options

The optimal exercise of American call options can be defined similarly as for put options, namely:

an optimal exercise time for the American call is a time t at which the call is in the money and $\widehat{C}(t, S(t), K, T) = (S(t) - K)_+$

However, assuming that (a) the underlying stock pays no dividends and (b) the risk-free rate r is positive, property (ii) and the put-call parity imply

$$\widehat{C}(t, S(t), K, T) \geq C(t, S(t), K, T) \geq S(t) - K\underbrace{e^{-r(T-t)}}_{<1} > \underbrace{S(t) - K}_{= (S(t)-K)_+ \text{ WHEN } S(t) > K}, \text{ for } t < T.$$

It follows that when the American call is in the money, i.e., when $S(t) > K$, there holds $\widehat{C}(t, S(t), K, T) > (S(t) - K)_+$, for $t < T$.

Therefore in an arbitrage-free market satisfying (a) and (b) *it is never optimal to exercise the American call prior to maturity* and so the option of early exercise of the American call is worthless.

This leads to the following last property for the price of options:

(x) When the underlying stock pays no dividend and $r > 0$, the value of the European call and the value of the American call with equal parameters are the same, i.e.,

$$\widehat{C}(t, S(t), K, T) = C(t, S(t), K, T).$$