Lecture_5

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Options and Mathematics: Lecture 5

November 10, 2020

Exercises

Exercise 1.9

Assume that at time t it is known that the underlying stock will pay the dividend $D < S(t_0)$ at time $t_0 \in (t, T)$. Prove the following variant of the put-call parity for $t < t_0$: (BEFORE THE DIVIDEND IS PAYED)

$$C(t, S(t), K, T) - P(t, S(t), K, T) = S(t) - KB(t, T) - DB(t, t_0).$$

PUT-CALL PARITY WITHOUT DIVIDENDS

$$C - P = S - KB$$

MOVE ALL TERMS ON THE RIGHT:

$$V_A(t) = S(t) - KB(t,T) - DB(t,t_0) - C(t, S(t), K, T) + P(t, S(t), K, T) = 0$$

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In the following exercises it is assumed that the arbitrage-free principle holds and that the stock pays no dividend.

Exercise 1.12

Consider the European derivative $\mathcal{U}$ with maturity time T and pay-off Y given by

CALL PUT

$$Y = \min[(S(T) - K_1)_+, (K_2 - S(T))_+],$$

where $K_2 > K_1$ and $(x)_+ = max(0, x)$. Draw the graph of the pay-off function of the derivative.

Find a constant portfolio consisting of European calls expiring at time T which **replicates** the value of $\mathcal{U}$, i.e., whose value at any time $t \leq T$ equals the value of $\mathcal{U}$.

PAY-OFF $\mathcal{U}_1 = (S(T) - K_1)_+$

PAY-OFF $\mathcal{U}_2 = S(T)\, H(S(T) - K_2)$

Exercise 1.14 (?)

Let $\mathcal{U}_1$ be a call stock option with strike K_1 and maturity T and $\mathcal{U}_2$ the physically-settled digital call option on the same stock with strike K_2 and maturity T. Decide whether the following statements are true or false and explain your answer.

TRUE (a) If $K_2 \leq K_1$, the value of $\mathcal{U}_2$ is greater or equal than the value of $\mathcal{U}_1$ for all $t < T$;

FALSE (b) If $K_2 > K_1$, the value of $\mathcal{U}_1$ is greater or equal than the value of $\mathcal{U}_2$ for all $t < T$.

HINT: DRAW THE GRAPHS OF THE PAY-OFFS IN CASE (a) AND (b)

Exercise 1.16

Consider the European derivative with pay-off Y at maturity T and the derivative with pay-off $Z = \Pi_Y(t_*)$ at maturity $t_* < T$. Show that $\Pi_Z(t) = \Pi_Y(t)$, $t \in [0, t_*]$

Exercise 1.17 (Chooser option)

Let $T_2 > T_1$. A chooser option with maturity T_1 is a contract which gives to the buyer the right to choose at time T_1 whether the derivative transforms (at zero cost) into a call or a put option with strike K and maturity T_2.

Write down the pay-off Y of the chooser option.

Let $r \in \mathbb{R}$ be constant. Show that the value of the chooser option is given by the formula

$$\Pi_Y(t) = C(t, S(t), K, T_2) + P(t, S(t), Ke^{-r(T_2-T_1)}, T_1), \quad t \leq T_1.$$

HINT: You need the result of Exercise 1.16 and the put-call parity.

Exercise 1.28

Find, if possible, constant portfolios consisting of European calls and/or puts that replicate the European derivatives with maturity T and pay-off Y depicted in the following figure.

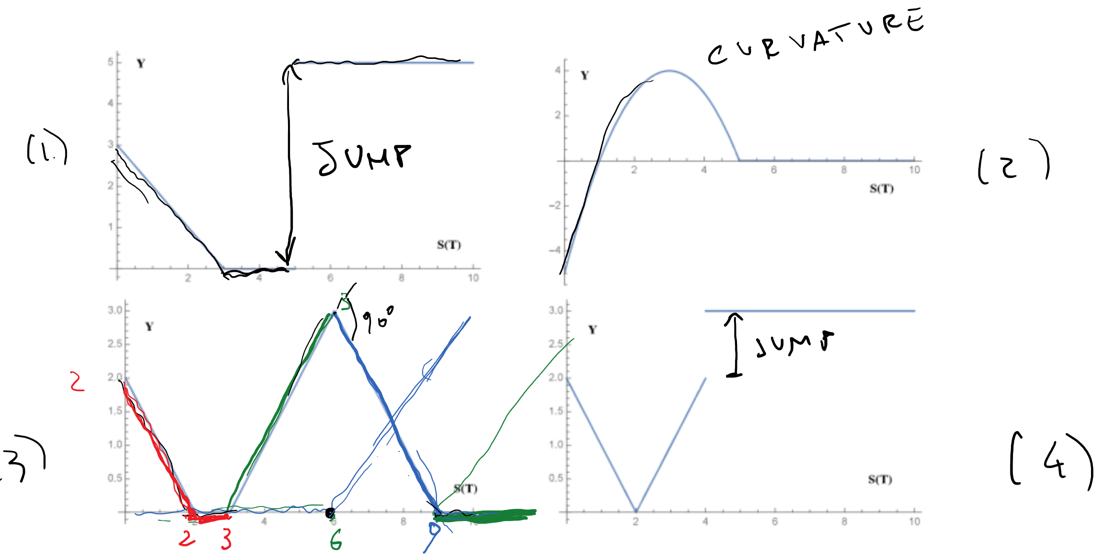


Figure 1: Remark: For $S(T) > 10$ the pay-off is identically zero

THE GOAL IS TO WRITE THE PAY-OFF AS A LINEAR COMBINATION OF PAY-OFFS OF CALL OPTIONS $(S(T)-K_1)_+$ AND PUT OPTIONS $(K_2-S(T))_+$.

(3) $(2-S(T))_+ + (S(T)-3)_+ - 2(S(T)-6)_+ + (S(T)-9)_+$

BY THEOREM 1.1 (b) WITH $C_A \equiv 0$,

$$\Pi_Y(t) = P(t,S(t),2,T) + C(t,S(t),3,T) - 2C(t,S(t),6,T) + C(t,S(t),9,T)$$

$r = \frac{d}{ds} \log B(s)$

$P(t, S(t), K, T) \longrightarrow 0$

WHEN $T \to \infty$

$\Rightarrow B(t) = B(0)\, e^{rt}$

AND SO IN PARTICULAR, IF $r > 0$ THEN $B(t)$ IS INCREASING IN TIME

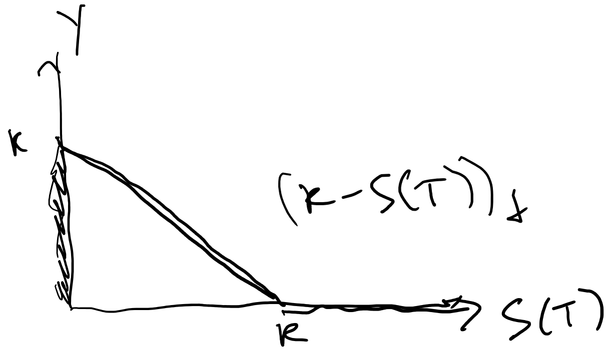
Exercise 1.30 (?) ASSUME $r > 0$

Decide whether the following statements are true and motivate your answer:

(INCREASING)

FALSE (a) The value of the European put option is non-decreasing with maturity;

TRUE (b) The value of the American put option is non-decreasing with maturity.



MAX PAY-OFF OF A PUT OPTION (EUROPEAN OR AMERICAN) IS $\underline{\underline{K}}$

$Y(t) = (K - S(t))_+$

Solution Exercise 1.9

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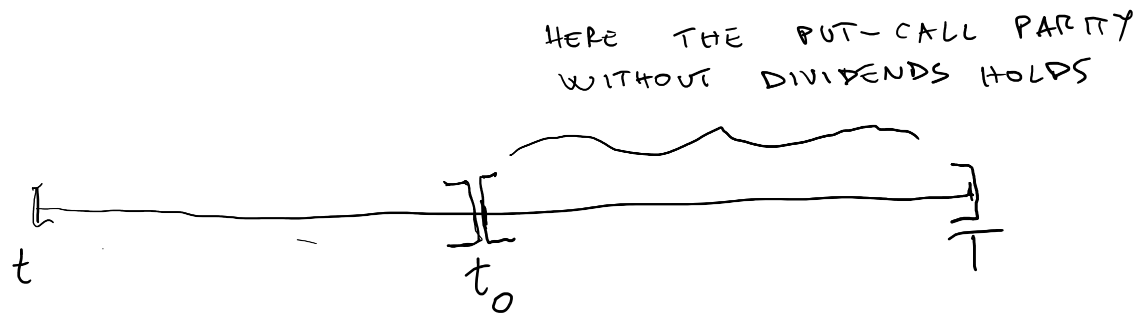

DIVIDEND IS PAYED AT TIME t_0

$$(I) \quad C(t, S(t), K, T) - P(t, S(t), K, T) = S(t) - K\,B(t,T)$$

FOR $t \in [t_0, T]$

$A = ($ 1 SHARE OF THE STOCK, $-K$ SHARES OF ZCB WITH MATURITY T, $-D$ SHARES OF ZCB WITH MATURITY t_0, -1 SHARE OF THE CALL, 1 SHARE OF THE PUT$)$

$$V_A(t) = S(t) - K\,B(t,T) - D\,B(t,t_0) - C(t, S(t), K, T) + P(t, S(t), K, T)$$

WANT TO PROVE THAT $\boxed{V_A(t) = 0}$ FOR $t < t_0$

THE VALUE OF A AT TIME t_0 IS

$$V_A(t_0) = \underline{S(t_0) - K\,B(t_0,T)} - D \underline{- C(t_0, S(t_0), K, T)} \underline{+ P(t_0, S(t_0), K, T)}$$

BY (I) ABOVE EVALUATED AT $t = t_0$, THE RED TERMS SUM UP TO ZERO. HENCE $V_A(t_0) = -D$. HENCE, BY THEOREM 1.1 (b) WITH $T = t_0$ AND $C_A = D$, WE CONCLUDE THAT $V_A(t) = 0$, FOR $t < t_0$ ∎

Solution to Exercise 1.12

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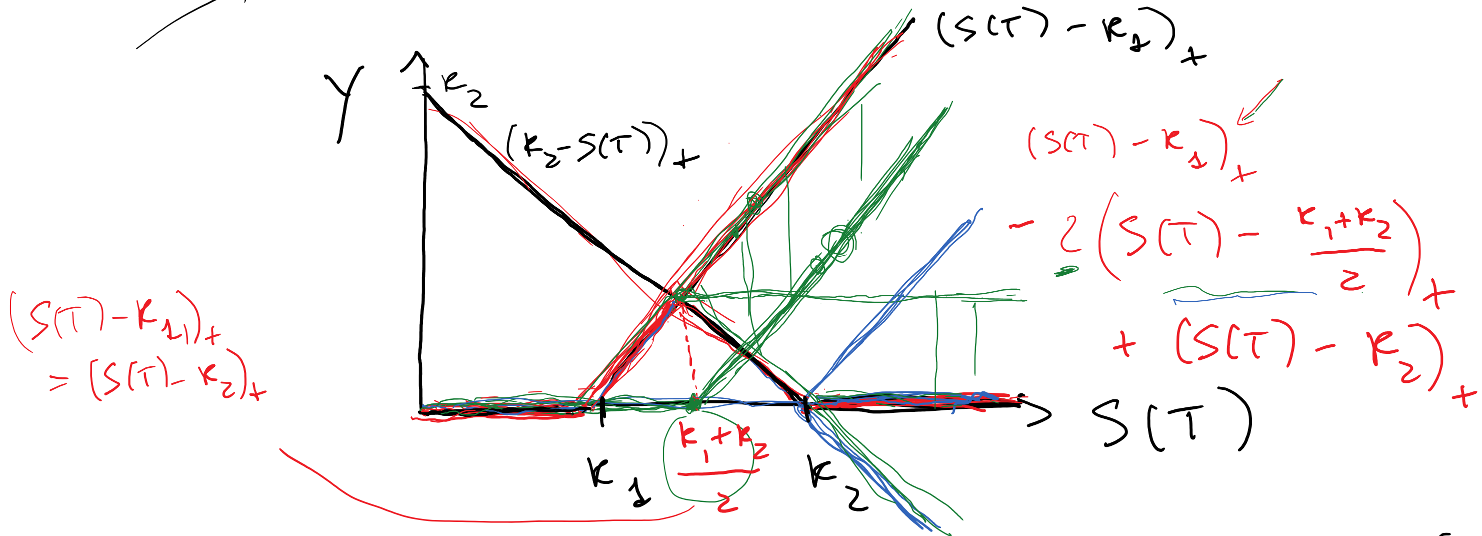
$$Y = \min\left((S(T)-K_1)_+ , (K_2 - S(T))_+ \right)$$



LET B_1 BE A CONSTANT PORTFOLIO WITH 1 SHARE OF $\mathcal{U}$.

GOAL: FIND A PORTFOLIO B_2 WITH CALL AND PUT OPTIONS SUCH THAT $V_{B_2}(t) = V_{B_1}(t)$, OR EQUIVALENTLY $V_{B_2}(t) - V_{B_1}(t) = V_{B_2-B_1}(t) = 0$

SINCE B_1 AND B_2 ARE SELF-FINANCING, THEN BY THEOREM 1.1(b), IT IS ENOUGH TO FIND B_2 SUCH THAT $V_{B_2-B_1}(T) = 0$, THAT IS

$$V_{B_2}(T) = V_{B_1}(T) = Y$$

SO, WE WANT TO EXPRESS Y AS A LINEAR COMBINATION OF PAY-OFFS OF CALL ~~AND/OR PUT~~ OPTIONS. FROM THE PICTURE ABOVE YOU CAN SEE THAT

YOU CAN SEE THAT

$$Y = (S(T) - K_1)_+ - 2\left(S(T) - \frac{K_1 + K_2}{2}\right)_+ + (S(T) - K_2)_+$$

HENCE $\mathcal{U}$ IS REPLICATED BY A PORTFOLIO WITH 1 SHARE OF THE CALL WITH STRIKE K_1, -2 SHARES OF THE CALL WITH STRIKE $\frac{K_1+K_2}{2}$ AN 1 SHARE OF THE CALL WITH STRIKE K_2.

Solution to Exercise 1.28

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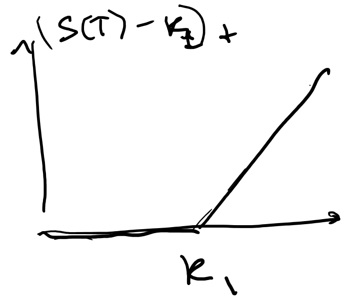
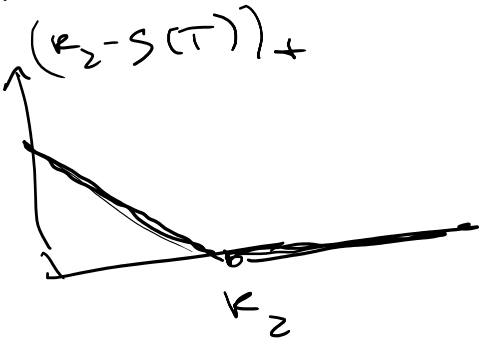
(1) NOT POSSIBLE! BECAUSE $(S(T)-K_1)_+$ AND $(K_2-S(T))_+$ ARE CONTINUOS FUNCTIONS:





AND SO ALSO ANY LINEAR COMBINATION OF THESE FUNCTIONS MUST BE CONTINUOS!

(4) IMPOSSIBLE FOR THE SAME REASON AS (1)

(2) IS NOT POSSIBLE BECAUSE ANY LINEAR COMBINATION OF CALL/PUT PAY-OFFS MUST BE PIECEWISE LINEAR (CANNOT HAVE CURVATURE!)

(3) IS POSSIBLE:

$$Y = (2-S(T))_+ + (S(T)-3)_+ - 2(S(T)-6)_+ + (S(T)-9)_+$$

Solution of Exercise 1.16

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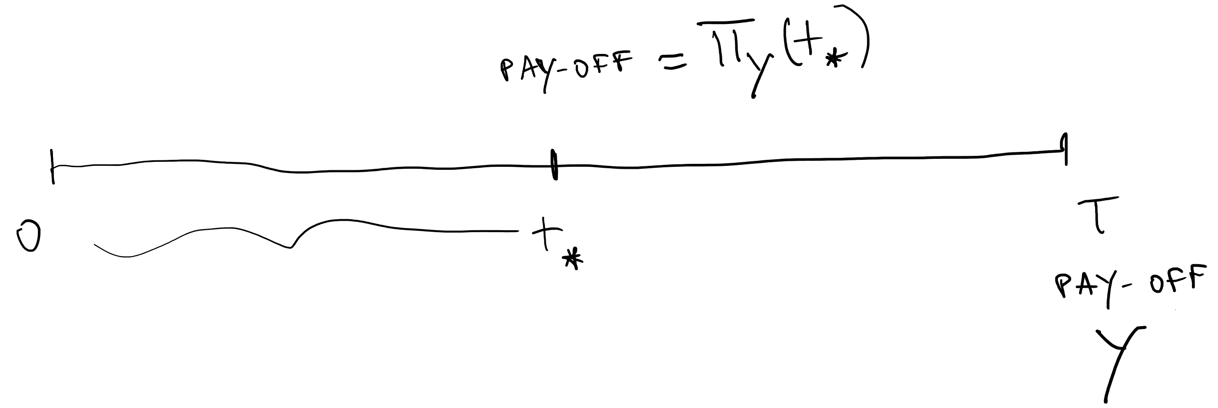

DENOTE $\Pi_Y(t)$ THE VALUE AT $t \in [0, T]$ OF THE DERIVATIVE WITH PAY-OFF AND MATURITY T. CALL $\mathcal{U}_1$ THIS DERIVATIVE

THE DERIVATIVE $\mathcal{U}_2$ EXPIRES AT $t_* < T$ WITH PAY-OFF $Z = \Pi_Y(t_*)$. SHOW THAT $\Pi_Z(t) = \Pi_Y(t)$ FOR $t < t_*$

SOLUTION: CONSIDER A PORTFOLIO A WITH 1 SHARE OF $\mathcal{U}_1$ AND -1 SHARE OF $\mathcal{U}_2$. THEN

$$V_A(t_*) = \Pi_Y(t_*) - \Pi_Z(t_*) = \Pi_Y(t_*) - Z$$
$$= \Pi_Y(t_*) - \Pi_Y(t_*) = 0$$

HENCE, BY THEOREM 1.1(b) WITH $T = t_*$ AND $C_A = 0$ (A is SELF-FINANCING), WE HAVE

$$V_A(t) = 0 \quad \text{FOR ALL } 0 \leq t \leq t_*$$

Solution to Exercise 1.17

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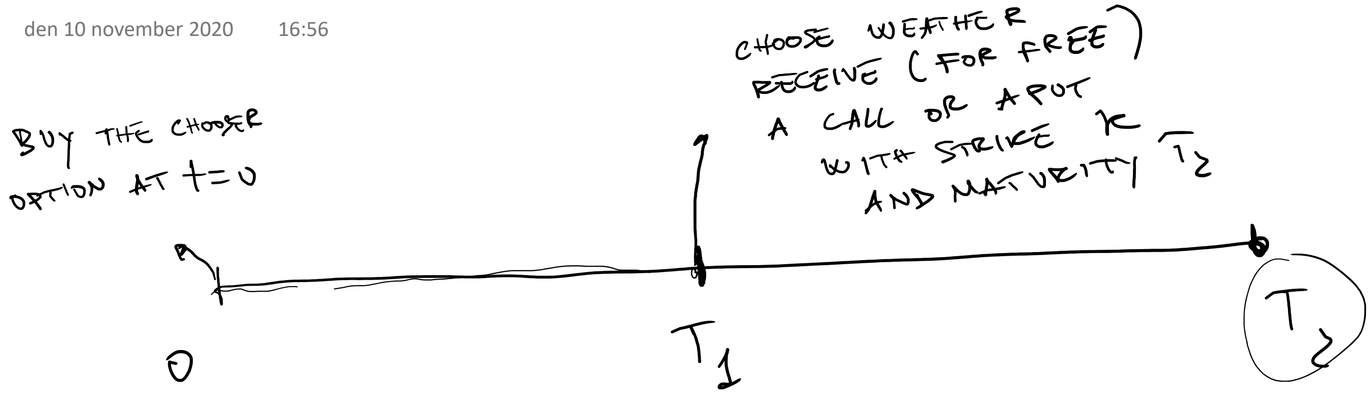

PAY-OFF OF THE CHOOSER OPTION

$$= \max\big(\underbrace{C(T_1, S(T_1), K, T_2)}_{a}, \underbrace{P(T_1, S(T_1), K, T_2)}_{b}\big)$$

$$= Y$$

USE THE IDENITY $\max(a,b) = a + \max(0, b-a)$

HENCE, $Y = C(T_1) + \max\big(0, P(T_1) - C(T_1)\big)$

$$\underset{\text{PUT CALL PARITY}}{=} C(T_1) + \underline{\max\big(0, K e^{-r(T_2-T_1)} - S(T_1)\big)}$$

$$e^{-r(T_2-T_1)} = B(T_1, T_2)$$

$$= Z + V \quad \text{WHERE } Z = C(T_1)$$

AND $V = \max\big(0, K e^{-r(T_2-T_1)} - S(T_1)\big)$

WHICH IS THE PAY OF PUT WITH STRIKE $K e^{-r(T_2-T_1)}$

$$\Pi_Y(t) = \Pi_Z(t) + \Pi_V(t) = \underline{\Pi_Z(t)} + P\big(t, S(t), K e^{-r(T_2-T_1)}, T_1\big)$$

$$\Pi_Z(t) = C(t, S(t), K, T_2) \quad \text{BY EXERCISE 1.16}$$

FOR ALL $t < T_1$

$$\Pi_Y(t) = \underbrace{C(t, S(t), K, T_2)}\; + \; P(t, S(t), \underline{Ke^{-r(T_2 - T_1)}}, T_1)$$

Solution to Exercise 1.14(a)

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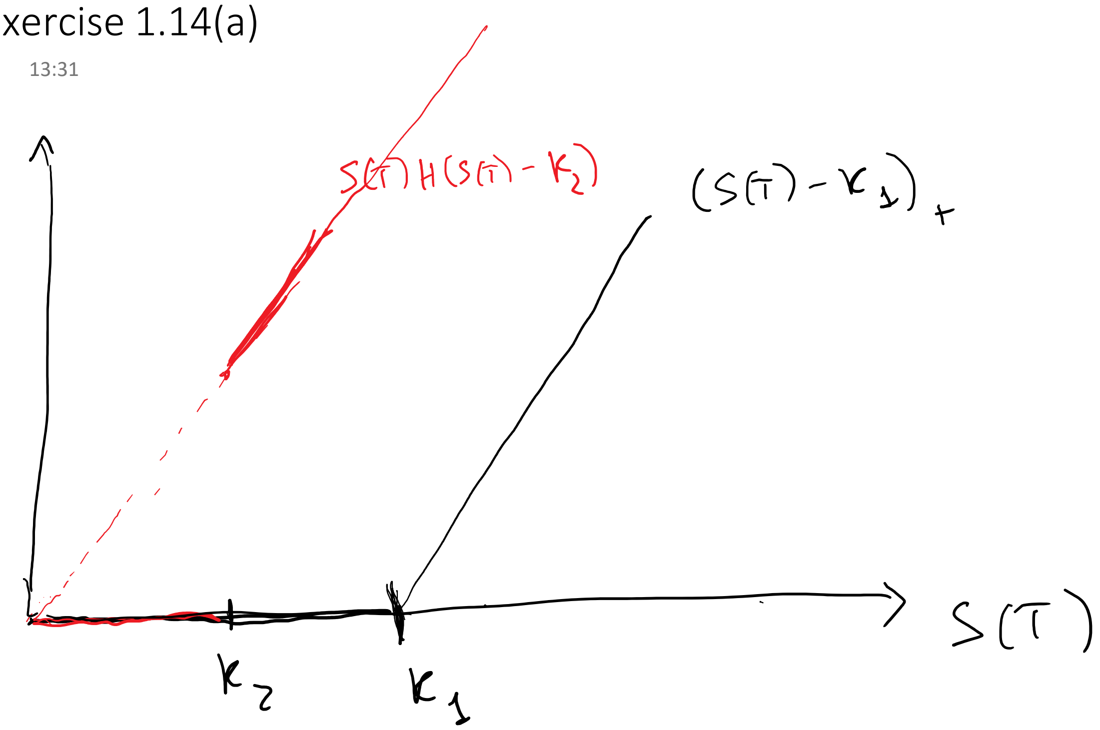
(a)

TRUE



IF THE CALL OPTION EXPIRES IN THE MONEY, SO DOES THE DIGITAL OPTION AND IN THIS CASE THE PAY OF $\mathcal{U}_2$ IS LARGER THAN $\mathcal{U}_1$

Solution to Exercise 1.14(b)

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(b)