Lecture_6

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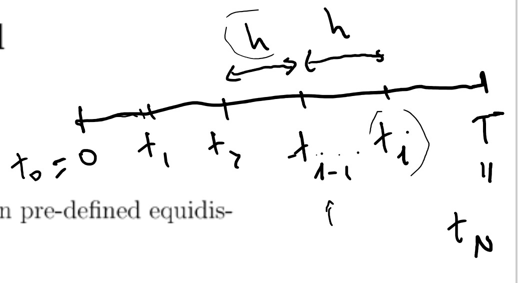
Options and Mathematics: Lecture 6

November 11, 2020

The Binomial Model



The binomial stock price

The binomial stock price can only change at some given pre-defined equidistant times:

$$0 = t_0 < t_1 < t_2 < \cdots < t_N = T, \quad t_i - t_{i-1} = h > 0, \; i = 1, \ldots, N.$$

Moreover the price of the stock at time t_i depends only on the price at time t_{i-1} and the result of "tossing a coin", namely given

$$u, d \in \mathbb{R}, \quad u > d, \quad \text{and} \quad p \in (0,1),$$

we assume

STOCK PRICE GOES UP AT TIME t_i

$$S(t_i) = \begin{cases} S(t_{i-1})e^u, & \text{with probability } p_u = p, \\ S(t_{i-1})e^d, & \text{with probability } p_d = 1 - p, \end{cases}$$

STOCK PRICE GOES DOWN AT TIME t_i

for all $i = 1, \ldots, N$.

$p_u + p_d = 1$

$(p_u, p_d) = (p, 1-p)$

Terminology

- The pair (p_u, p_d) is called **physical** (or **real-world**) **probability vector**, to distinguish it from the **risk-neutral probability vector** introduced later.
- If $S(t_i) = S(t_{i-1})e^u$ we say that the price goes up at time t_i
- If $S(t_i) = S(t_{i-1})e^d$ we say that the price goes up at time t_i

REMARK: STRICTLY TRUE IF $u > 0$ AND $d < 0$

Definition 2.1

The **instantaneous mean of log-return** $\alpha \in \mathbb{R}$ and the **instantaneous volatility** $\sigma > 0$ of the binomial stock price are defined respectively by

$$\boxed{\alpha = \frac{1}{h}[pu + (1-p)d], \quad \sigma = \frac{u-d}{\sqrt{h}}\sqrt{p(1-p)}}$$

The quantity σ^2 is called **instantaneous variance**.

Inverting the previous equation we have

$$\boxed{u = \alpha h + \sigma\sqrt{\frac{1-p}{p}}\sqrt{h}, \quad d = \alpha h - \sigma\sqrt{\frac{p}{1-p}}\sqrt{h}}$$

Hence we may formulate the binomial stock price by either

(i) specifying the parameters u, d, p, h, T or by

(ii) specifying the parameters α, σ, p, h, T and computing u, d with the formulas above.

The formulation (ii) is the most common in the applications and will be used for the numerical implementation of the binomial model.

The formulation (i) is more suitable for hand calculations and will be used for the theoretical analysis of the model.

$t \in$ $S(0)$ $S(1)$ $S(2)$ $S(3)$ ...

$t_0 = 0$ $t_1 = 1$ 2 3 $T = N$

For the theoretical analysis we also assume that $h = 1$ and so

$$t_1 = 1, \quad t_2 = 2, \quad \ldots, \quad t_N = T = N.$$

It is convenient to denote

$$\boxed{\mathcal{I} = \{1, \ldots, N\}}$$

$t \in \mathcal{I}$

Hence, from now on, for the theoretical analysis of the binomial model we assume that the binomial stock price is determined by the rule $S(0) = S_0 > 0$ and

KNOWN

$$\boxed{S(t) = \begin{cases} S(t-1)e^u, & \text{with probability } p \\ S(t-1)e^d, & \text{with probability } 1-p \end{cases}, \quad t \in \mathcal{I}}$$

Paths of the binomial stock price

Each possible sequence $(S(0), S(1), \ldots, S(N))$ of stock prices determined by the binomial model is called a **path** of the stock price.

There exists 2^N possible paths of the stock price in a N-period model. Letting

$$\boxed{\{u,d\}^N = \{x = (x_1, x_2, \ldots, x_N) : x_t = u \text{ or } x_t = d, \ t \in \mathcal{I}\}}$$

we obtain a unique path of the stock price $(S(0), S(1), \ldots, S(N))$ for each $x \in \{u,d\}^N$.

Example:

$x = (u, d, u) \in \{u, d\}^3$

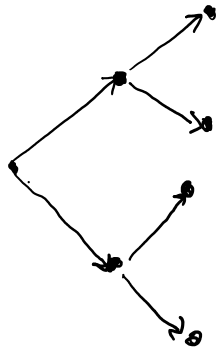
$x = (u, d, u)$ in the 3-period model corresponds to the path

$$S_0 \to S(1) = S_0 e^u \to S(2) = S(1)e^d = S_0 e^{u+d} \to S(3) = S(2)e^u = S_0 e^{2u+d}.$$

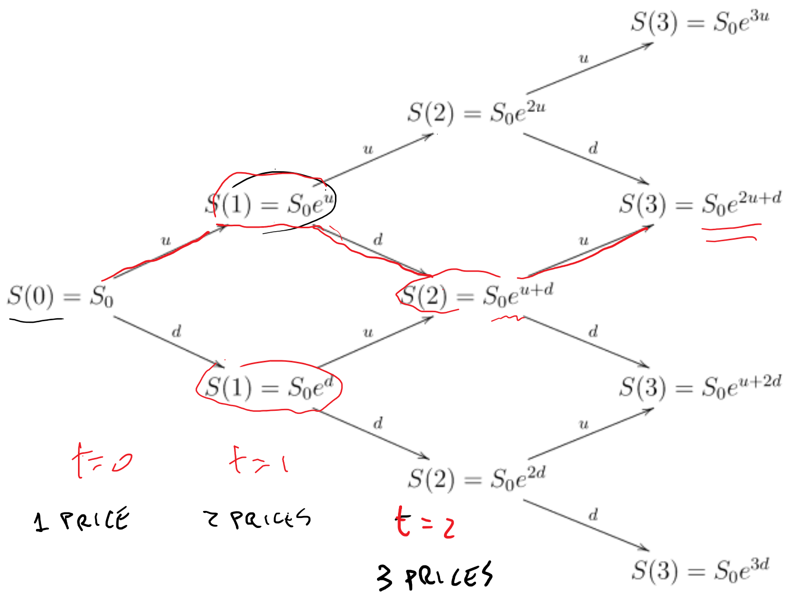
3

$x = (u, u, d, d, d) \in \{u, d\}^5$ $\quad S(5) = S_0 e^{2u+3d}$

In general the possible paths of the stock price for the 3-period model can be represented as



NON-RECOMBINING AT $t=2$



$x = (u, d, u)$

THERE ARE $t+1$ POSSIBLE STOCK PRICES AT TIME t

which is an example of **recombining binomial tree**.

The admissible values for the binomial stock price $S(t)$ at time t are given by

$$S(t) \in \{S_0 e^{ku+(t-k)d},\ k = 0, \ldots, t\},$$

for all $t \in \mathcal{I}$, where k is the number of times that the price goes up up to the time t included.

In particular, at time t there are $t+1$ possible values for the stock price.

$x = (u, d, d, u, u) \in \{u, d\}^5$ $k = 3$

$k = 0, 1, 2, 3, 4, 5$

Definition 2.2

Given $x = (x_1, \ldots, x_N) \in \{u, d\}^N$, the binomial stock price at time $t \in \mathcal{I}$ $= \{1, 2, \ldots N\}$ along the path x is given by

$$S(t, x_1, \ldots, x_t) = S_0 \exp(x_1 + x_2 + \ldots x_t).$$

The vector $S^x = (S(0), S(1, x_1), S(2, x_1, x_2), \ldots, S(N, x_1, \ldots, x_N))$ is called the x-path of the binomial stock price.

Moreover we define the probability of the path S^x as

$$\mathbb{P}(S^x) = p^{N_u(x)}(1-p)^{N_d(x)},$$

where $N_u(x)$ is the number of u's in the sequence x and $N_d(x) = N - N_u(x)$ is the number of d's.

$x, y \in \{u, d\}^N$

The probability that the binomial stock price follows one of the two paths S^x, S^y is given by $\mathbb{P}(S^x) + \mathbb{P}(S^y)$, and similarly for any number of paths.

Example:

For $x = (u, d, d, d, u)$ in a 5-period binomial model, the x-path of the stock price is

$$S^x = (S(0), S(1, u), S(2, u, d), S(3, u, d, d), S(4, u, d, d, d), S(5, u, d, d, d, u))$$

$$= (S_0, S_0 e^u, S_0 e^{u+d}, S_0 e^{u+2d}, S_0 e^{u+3d}, S_0 e^{2u+3d})$$

$3 = 5 - 2$

and the probability that the stock follows this path is $\mathbb{P}(S^x) = p^2(1-p)^3$.

$y = (u, u, u, d, d)$ $\quad S^y = (S(0), S(1, u), S(2, u, u), S(3, u, u, u), S(4, u, u, u, d), S(5, u, u, u, d, d))$

$\mathbb{P}(S^y) = p^3 (1-p)^2 \qquad \mathbb{P}(S^x \text{ or } S^y) = \mathbb{P}(S^x) + \mathbb{P}(S^y)$

$= p^2(1-p)^3 + p^3(1-p)^2$

Exercise 2.2

Show that
$$\sum_{x\in\{u,d\}^N} \mathbb{P}(S^x) = 1.$$

HINT: You need the **binomial theorem**:
$$(a+b)^N = \sum_{k=0}^{N} \binom{N}{k} a^k b^{N-k}, \tag{1}$$

where $a, b > 0$ and
$$\binom{N}{k} = \frac{N!}{k!(N-k)!}$$

is the **binomial coefficient**.

SOLUTION:

$$\sum_{x\in\{u,d\}^N} \mathbb{P}(S^x) = \sum_{x\in\{u,d\}^N} p^{N_u(x)} (1-p)^{N_d(x)}$$

$$= \sum_{x\in\{u,d\}^N} p^{N_u(x)} (1-p)^{N-N_u(x)}$$

(u,u,d)

(u,d,u)

NUMBER OF PATHS WITH $N_u(x) = k$ IS GIVEN BY THE BINOMIAL COEFFICIEN $\binom{N}{k}$, $k = 0, \ldots N$

$$\sum_{x\in\{u,d\}^N} \mathbb{P}(S^x) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} = (p+1-p)^N = 1 \quad \blacksquare$$

$a = p$
$b = 1-p$ IN BINOMIAL THEO.

Binomial markets

A **binomial market** is a market that consists of

- one stock with price $S(t)$ given by the binomial model

$$S(t) = \begin{cases} S(t-1)e^{u}, & \text{with probability } p \\ S(t-1)e^{d}, & \text{with probability } 1-p \end{cases}, \quad t \in \mathcal{I}.$$

- One risk-free asset with value

$$\boxed{B(t) = B_0 e^{rt}, \quad t \in \mathcal{I}}$$

where $B_0 = B(0) > 0$ is the value of the risk-free asset at time $t = 0$ and $r \in \mathbb{R}$ is the continuously compounded risk-free rate.

We assume that r is constant (not necessarily positive), which is reasonable for short time investments ($T \lessapprox 1$ year).

The constants u, d, r, p are called **market parameters**.

EXAMPLE : $t=0$ PRESENT TIME, $S(t) = 10\$$ $t>0$ (IN THE FUTURE)

1\$ AT $t=0$ CORRESPONDS TO $1 \cdot e^{rt}$ \$ AT TIME t, BECAUSE WE CAN INVEST 1\$ TODAY ON THE MONEY MARKET AND RECEIVE RISK-FREE e^{rt} \$ AT TIME t.

Discounted price of the stock

The **discounted price** (at time $t = 0$) of the stock in a binomial market is defined by

$$\boxed{S^*(t) = e^{-rt}S(t)} \quad t \in I \quad = \frac{S(t)}{e^{rt}}$$

and has the following meaning:

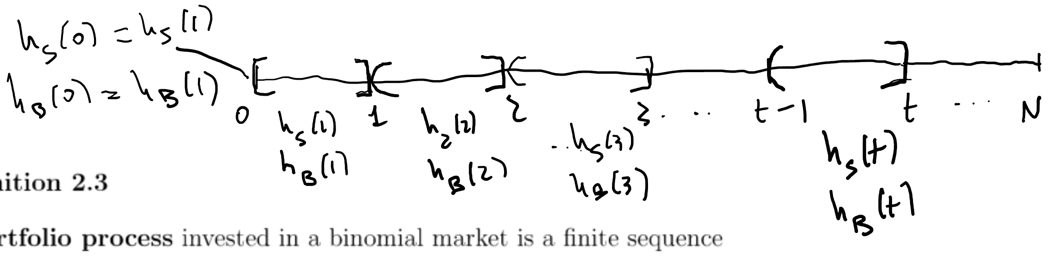
$S^*(t)$ is the amount that should be invested at time $t = 0$ on the risk-free asset in order that the value at time t of this investment replicates the value of the stock at time t.

$$B(0) = e^{-rt} S(t)$$

$$B(t) = B(0) e^{rt} = S(t)$$

Remarks:

- $S^*(t)$ is also called "today's value" of the stock (assuming $t = 0$ is the present time).
- If $r > 0$, i.e., if buying the risk-free asset ensures a positive return, then $S^*(t) < S(t)$. The discounted price of the stock measures the loss in the stock value due to the "time-devaluation" of money expressed by the ratio $B_0/B(t) = e^{-rt}$.
- In finance it is agreed that 1 dollar today ($t = 0$) is equivalent, in terms of purchasing power, to e^{rt} dollars at time t in the future, because one can invest the dollar in the money market today and receives (risk-free) e^{rt} dollars at time t. Hence $S^*(t) = S(t)/e^{rt}$ measures the value of the stock at the future time t relative to the purchasing value of 1 dollar at time t.



Definition 2.3

A **portfolio process** invested in a binomial market is a finite sequence

$$\{(h_S(t), h_B(t))\}_{t \in \mathcal{I}},$$

where $h_S(t) \in \mathbb{R}$ is the number of shares invested in the stock and $h_B(t) \in \mathbb{R}$ is the number of shares invested in the risk-free asset during the time interval $(t-1, t]$, $t \in \mathcal{I}$. The value of the portfolio process at time t is given by

$$\boxed{V(t) = h_S(t)S(t) + h_B(t)B(t)}$$

Remarks:

- The initial position of the investor is given by $(h_S(0), h_B(0))$. As portfolio positions held for one instant of time only are clearly meaningless, we may assume that

$$\boxed{h_S(0) = h_S(1), \quad h_B(0) = h_B(1)}$$

  i.e, $(h_S(1), h_B(1))$ is the investor position on the closed interval $[0, 1]$ (and not just in the semi-open interval $(0, 1]$).

- Recall that $h_S(t) > 0$ means that the investor has a long position on the stock in the interval $(t-1, t]$, while $h_S(t) < 0$ corresponds to a short position.

It is clear that the investor will change the position on the stock and the risk-free asset according to the path followed by the stock price, and so $(h_S(t), h_B(t))$ is in general path-dependent.

We are only interested in portfolio processes for which the position on the assets in the interval $(t-1, t]$ is chosen based on the information available at time $t-1$ and not on the uncertain future. By "information" at time t here we mean the knowledge of the path $(x_1, \dots, x_t)$ followed by the stock price up to time t.

Definition 2.4

POSITION ON THE INTERVAL $(t-1, t]$

A portfolio process $\{(h_S(t), h_B(t)\}_{t\in\mathcal{I}}$ is called **predictable** if $h_S(t)$ and $h_B(t)$ depend only on the path $(x_1, \dots, x_{t-1})$ followed by the stock price up to time $t-1$.

When we want to emphasize the dependence of a predictable portfolio position on the path of the stock price we shall write

$$\boxed{h_S(t) = h_S(t, \underbrace{x_1, \dots, x_{t-1}}_{t}), \quad h_B(t) = h_B(t, x_1, \dots, x_{t-1})}$$

Likewise we write

$$V(t) = h_S(t) S(t) + h_B(t) \underbrace{B(t)}_{B_0 e^{rt}}$$

$$\boxed{V(t, x_1, \dots, x_t) = h_S(t, x_1, \dots, x_{t-1}) S(t, x_1, \dots, x_t) + h_B(t, x_1, \dots, x_{t-1}) B(t)}$$

for the value of predictable portfolio processes.

$$S(t) = S_0 e^{x_1 + x_2 + \dots + x_{t-1} + x_t}$$

$(h_S(t), h_B(t)) = (h_S(t, x_1 \dots x_{t-1}),$
$h_B(t, x_1 \dots x_{t-1}))$

$t-1$ $\qquad$ t

$S(t-1) = S(t-1, x_1, x_2, \dots x_{t-1})$

Example:

A portfolio process is predictable when the position $(h_S(t), h_B(t))$ in the interval $(t-1, t]$ is a **deterministic function** of the stock price at time $t-1$, that is

$$\boxed{(h_S(t), h_B(t)) = H_t(S(t-1)), \quad H_t : (0, \infty) \to \mathbb{R}^2, \quad t \in \mathcal{I}}$$

$(h_S(3), h_B(3)) = H_3(S(2))$

All portfolio processes considered in this course have this special form and thus they are predictable

In this case the portfolio process can be represented with a recombining binomial tree, as in the following example for $N = 3$:

$(h_S(3), h_B(3)) = H_3(S_0 e^{2u})$

u

$(h_S(2), h_B(2)) = H_2(S_0 e^u)$

d

$(h_S(1), h_B(1)) = H_1(S_0)$ $\qquad$ $(h_S(3), h_B(3)) = H_3(S_0 e^{u+d})$

d $\qquad$ u

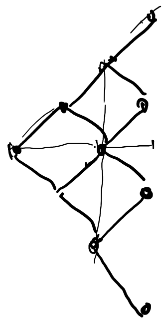
$(h_S(2), h_B(2)) = H_2(S_0 e^d)$

d

$(h_S(3), h_B(3)) = H_3(S_0 e^{2d})$

$(h_S(2), h_B(2)) = H_2(S(1))$

N = 3 BINOMIAL TREE



$[0, 1]$ $\qquad$ $(1, 2]$ $\qquad$ $(2, 3]$

Since $(h_S(0), h_B(0)) = (h_S(1), h_B(1))$, the binomial tree for a portfolio process in an N-period binomial model has only $N - 1$ periods.