Lecture_7

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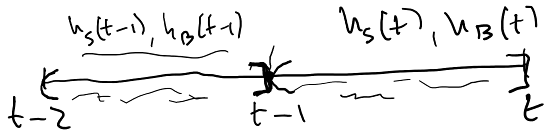
Options and Mathematics: Lecture 7

November 12, 2020

The Binomial Model

Self-financing portfolio

A portfolio process in the binomial market is self-financing if purchasing more shares of one asset is possible only by selling shares of the other asset for an equivalent value (and not by infusing new cash into the portfolio), and, conversely, if any cash obtained by selling shares of one asset is immediately re-invested to buy shares of the other asset (and not withdrawn from the portfolio).



Let $(h_S(t-1), h_B(t-1))$ be the investor position on the stock and the risk-free asset during the time interval $(t-2, t-1]$.

Assume that the investor change the position on the assets at time $t-1$ and let $(h_S(t), h_B(t))$ be the new position in the interval $(t-1, t]$.

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$$\rightarrow V(t-1) = h_S(t-1)S(t-1) + h_B(t-1)B(t-1) \equiv$$ PORTFOLIO VALUE WHEN THE POSITION IS CHANGED A TIME $t-1$

$$V'(t-1) = h_S(t)S(t-1) + h_B(t)B(t-1) \equiv$$ PORTFOLIO VALUE IMMEDIATELY AFTER THE POSITION IS CHANGED AT TIME $t-1$

$$V(t) = h_S(t)S(t) + h_B(t)B(t) \leftarrow$$ PORTFOLIO VALUE IN THE INTERVAL

$V(t) = h_S(t)S(t) + h_B(t)B(t)$ ← PORTFOLIO VALUE IN THE INTERVAL $(t-1, t]$

The value of the portfolio process at time $t-1$ is

$$V(t-1) = h_S(t-1)S(t-1) + h_B(t-1)B(t-1),$$ → AT TIME $t-1$

while the value "immediately after" changing the position at time $t-1$ is

$$V'(t-1) = h_S(t)S(t-1) + h_B(t)B(t-1).$$ → IMMEDIATELY AFTER TIME $t-1$

The difference $V'(t-1) - V(t-1)$, if not zero, corresponds to cash withdrawn or added to the portfolio as a result of the change in the position on the assets.

In a self-financing portfolio this difference must be zero. We thus must have $V'(t-1) = V(t-1)$, which leads to the following definition.

Definition A portfolio process $\{(h_S(t), h_B(t))\}_{t\in\mathcal{I}}$ invested in a binomial market is said to be **self-financing** if

$$\boxed{h_S(t)S(t-1) + h_B(t)B(t-1) = h_S(t-1)S(t-1) + h_B(t-1)B(t-1)}$$ ⟺ $V'(t-1) = V(t-1)$

holds for all $t \in \mathcal{I}$.

Exercise 2.4

Consider a 3-period binomial model with the following parameters:

$$u = \log\frac{5}{4}, \quad d = \log\frac{1}{2}, \quad r = \log\frac{3}{4}, \quad S(0) = B(0) = 64$$ $N = 3$

and the portfolio process in this market given by

$[0,1]$ $h_S(1) = 1, \quad h_B(1) = -1,$

$(1,2]$ $h_S(2,u) = -1, \quad h_B(2,u) = 7/3, \quad h_S(2,d) = 7/2, \quad h_B(2,d) = -8/3,$

$(2,3]$ $h_S(3,u,u) = 1, \quad h_B(3,u,u) = -29/9, \quad h_S(3,d,d) = 49/4, \quad h_B(3,d,d) = -59/9,$
$h_S(3,u,d) = h_S(3,d,u) = -7/2, \quad h_B(3,u,d) = h_B(3,d,u) = 46/9.$

Show that this portfolio is self-financing. Compute the value of the portfolio process along any possible path and represent the result with a binomial tree.

THE BINOMIAL TREE OF THE PORTFOLIO IS RECOMBINING

you

IS (q_u, q_d) A PROBABILITY VECTOR?
IN GENERAL, NO, BECAUSE WE DON'T KNOW IF $q = q_u \in (0,1)$

$p_u = p$ $\quad p \in (0,1)$

$p_d = 1 - p$

PHYSICAL PROBABILITY

Notation

We define the parameters q_u, q_d as

$$\boxed{q_u = q, \quad q_d = 1 - q, \quad \text{where } q = \frac{e^r - e^d}{e^u - e^d}}$$

Note that (q_u, q_d) is the unique solution of the linear system

$$q_u + q_d = 1, \quad q_u e^u + q_d e^d = e^r.$$

Given a self-financing portfolio process, we denote

$V(t) = h_S(t)S(t) + h_B(t)B(t)$

$B(t) = B(t-1)e^r$

$$V^u(t) = h_S(t)S(t-1)e^u + h_B(t)B(t-1)e^r,$$

$S(t) = \begin{cases} S(t-1)e^u \\ S(t-1)e^d \end{cases}$

which is the value of the portfolio at time t assuming that the stock price goes up at time t, and

$$V^d(t) = h_S(t)S(t-1)e^d + h_B(t)B(t-1)e^r,$$

which is the value of the portfolio at time t, assuming that the stock price goes down at time t.

Note that, SINCE $V(t) = V(t, x_1, x_2, \ldots x_t)$

$$V^u(t) = V^u(t, x_1, \ldots, x_{t-1}) = V(t, x_1, \ldots, x_{t-1}, u),$$

and similarly for $V^d(t)$. $= V(t, x_1, x_2, \ldots x_{t-1}, d)$

We now prove two identities satisfied by the value of self-financing portfolios.

$V^u(t+1) = h_S(t+1)S(t)e^u + h_B(t+1)B(t)e^r$

SELF FINANCING PROPERTY AT TIME $t+1$:

$$h_S(t+1)S(t) + h_B(t+1)B(t) = h_S(t)S(t) + h_B(t)B(t) = V(t)$$

Theorem 2.1 (recurrence formula for the value of self-financing portfolios)

The value $V(t)$ of a self-financing portfolio process $\{(h_S(t), h_B(t))\}_{t\in\mathcal{I}}$ satisfies the following recurrence formula:

$$V(t) = e^{-r}[q_u V^u(t+1) + q_d V^d(t+1)], \quad \text{for } t = 0, \dots, N-1.$$

Proof: Using the definition of $V^u(t), V^d(t)$, we have

$$\begin{aligned}
e^{-r}[q_u V^u(t+1) + q_d V^d(t+1)] &= e^{-r}[q_u \overbrace{(h_S(t+1)S(t)e^u + h_B(t+1)B(t)e^r)}^{V^u(t+1)} \\
&\quad + q_d (h_S(t+1)S(t)e^d + h_B(t+1)B(t)e^r)] \\
&= e^{-r}[h_S(t+1)S(t)(q_u e^u + q_d e^d) + h_B(t+1)B(t)e^r(q_u + q_d)] \\
&= h_S(t+1)S(t) + h_B(t+1)B(t), \\
&= V(t)
\end{aligned}$$

By definition of self-financing portfolio, the last member of equals $V(t)$, and so the theorem is proved.

Theorem 2.$\cancel{6}$ 2

Let $\{(h_S(t), h_B(t))\}_{t\in\mathcal{I}}$ be a self-financing portfolio process with value $V(N, x)$ at time $t = N$ along the path x. The portfolio value $V(t)$ at earlier times satisfies

$$\boxed{V(t) = e^{-r(N-t)} \sum_{(x_{t+1},\dots,x_N)\in\{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} V(N, x), \quad \text{for } t = 0, \dots, N-1.}$$

each of these is either q_u or q_d

In particular, at time $t = 0$ we have

$$\boxed{V(0) = e^{-rN} \sum_{x\in\{u,d\}^N} q_u^{N_u(x)} q_d^{N_d(x)} V(N, x).}$$

$N_d(x) = N - N_u(x)$

IF I KNOW $V(N,x)$, THEN I CAN COMPUTE $V(0)$

Exercise 2.5

Show that the self-financing portfolio process in Exercise 2.4 satisfies the formulas in the previous theorems. HINT: Use the binomial tree of $V(t)$ derived in Exercise 2.4.

Solution of Exercise 2.4

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$$S(0) = 64, \quad u = \log\frac{5}{4} \quad d = \log\frac{1}{2}, \quad r = \log\frac{3}{4}$$

$$S(t) = \begin{cases} S(t-1)\,\frac{5}{4} & \text{WITH PROB } p \\ S(t-1)\,\frac{1}{2} & \text{WITH PROB } 1-p \end{cases}$$

$$\begin{tikzcd}[column sep=small]
 & & & S(3)=125 \\
 & & S(2)=100 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & S(1)=80 \arrow[ur, "u"] \arrow[dr, "d"] & & S(3)=50 \\
S(0)=64 \arrow[ur, "u"] \arrow[dr, "d"] & & S(2)=40 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & S(1)=32 \arrow[ur, "u"] \arrow[dr, "d"] & & S(3)=20 \\
 & & S(2)=16 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & & S(3)=8
\end{tikzcd}$$

$$B(0) = 64, \quad B(1) = 64\cdot\frac{3}{4} = 48, \quad B(2) = 64\left(\frac{3}{4}\right)^2 = 36 = B(0)e^{2r}$$

$$B(3) = B(0)e^{3r} = B(0)(e^{r})^3 = 64\cdot\left(\frac{3}{4}\right)^3 = 27$$

Solution to Exercise 2.4(b)

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$$h_S(0) = h_S(1) \qquad h_B(0) = h_B(1)$$

$$\begin{tikzcd}
 & & \begin{matrix} h_S(3)=1 \\ h_B(3) = -29/9 \end{matrix} \\
 & \begin{matrix} h_S(2)=-1 \\ h_B(2) = 7/3 \end{matrix} \arrow[ur, "u"] \arrow[dr, "d"] & \\
\begin{matrix} h_S(1)=1 \\ h_B(1) = -1 \end{matrix} \arrow[ur, "u"] \arrow[dr, "d"] & & \begin{matrix} h_S(3)=-7/2 = h_S(3,u,d) = h_S(3,d,u) \\ h_B(3) = 46/9 \end{matrix} \\
 & \begin{matrix} h_S(2)=7/2 \\ h_B(2) = -8/3 \end{matrix} \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & \begin{matrix} h_S(3)=49/4 \\ h_B(3) = -59/9 \end{matrix}
\end{tikzcd}$$

SELF-FINANCING:

$$h_S(t)S(t-1) + h_B(t)B(t-1) = h_S(t-1)S(t-1) + h_B(t-1)B(t-1)$$

$t = 1, 2, 3$

1 ↳ ALWAYS TRUE

$\boxed{t=1}$

$$h_S(1)S(0) + h_B(1)B(0) = h_S(0)S(0) + h_B(0)B(0)$$

SINCE $h_S(1) = h_S(0)$, $h_B(1) = h_B(0)$, THEN THIS IS TRUE

$\boxed{t=2}$

$$h_S(2)S(1) + h_B(2)B(1) = h_S(1)S(1) + h_B(1)B(1)$$

u: $h_S(2,u)S(1,u) + h_B(2,u)B(1) = h_S(1)S(1,u) + h_B(1)B(1)$

THAT IS $-1\cdot 80 + \frac{7}{3}\cdot 48 = 1\cdot 80 + (-1)48$ i.e. $\underbrace{-80+112}_{32} = \underbrace{80-48}_{32}$ ✓

d: $h_S(2,d)S(1,d) + h_B(2,d)B(1) = h_S(1)S(1,d) + h_B(1)B(1)$

$$\frac{7}{2}\cdot 32 + \left(-\frac{8}{3}\right)48 = 1\cdot 32 + (-1)48$$

$$112 - 128 = 32 - 48$$
$$-16 = -16 \checkmark$$

$t = 3$

$$h_S(3)S(2) + h_B(3)B(2) = h_S(2)S(2) + h_B(2)B(2)$$

$$h_S(3,u,u)S(2,u,u) + h_B(3,u,u)B(2) = h_S(2,u)S(2,u,u) + h_B(2,u)B(2)$$
$$h_S(3,u,d)S(2,u,d) + h_B(3,u,d)B(2) = h_S(2,u)S(2,u,d) + h_B(2,u)B(2)$$
$$h_S(3,d,u)S(2,d,u) + h_B(3,d,u)B(2) = h_S(2,d)S(2,d,u) + h_B(2,d)B(2)$$
$$h_S(3,d,d)S(2,d,d) + h_B(3,d,d)B(2) = h_S(2,d)S(2,d,d) + h_B(2,d)B(2)$$

THESE EQUATIONS ARE VERIFIED

PORTFOLIO VALUES:

$$V(0) = h_S(0)S(0) + h_B(0)B(0) = h_S(1)S(0) + h_B(1)B(0)$$
$$= 1 \cdot 64 - 1 \cdot 64 = 0$$

$$V(1) = h_S(1)S(1) + h_B(1)B(1) \begin{cases} \xrightarrow{u} h_S(1)S(1,u) + h_B(1)B(1) = 32 \\ \xrightarrow{d} h_S(1)S(1,d) + h_B(1)B(1) = -16 \end{cases}$$

$$V(2) = h_S(2)S(2) + h_B(2)B(2) \begin{cases} \to V(2,u,u) = h_S(2,u)S(2,u,u) + h_B(2,u)B(2) = -16 \\ \to V(2,u,d) = V(2,d,u) = 44 \\ \to V(2,d,d) = -40 \end{cases}$$

IN THE BOOK IT IS WRITTE -6 (TYPO)

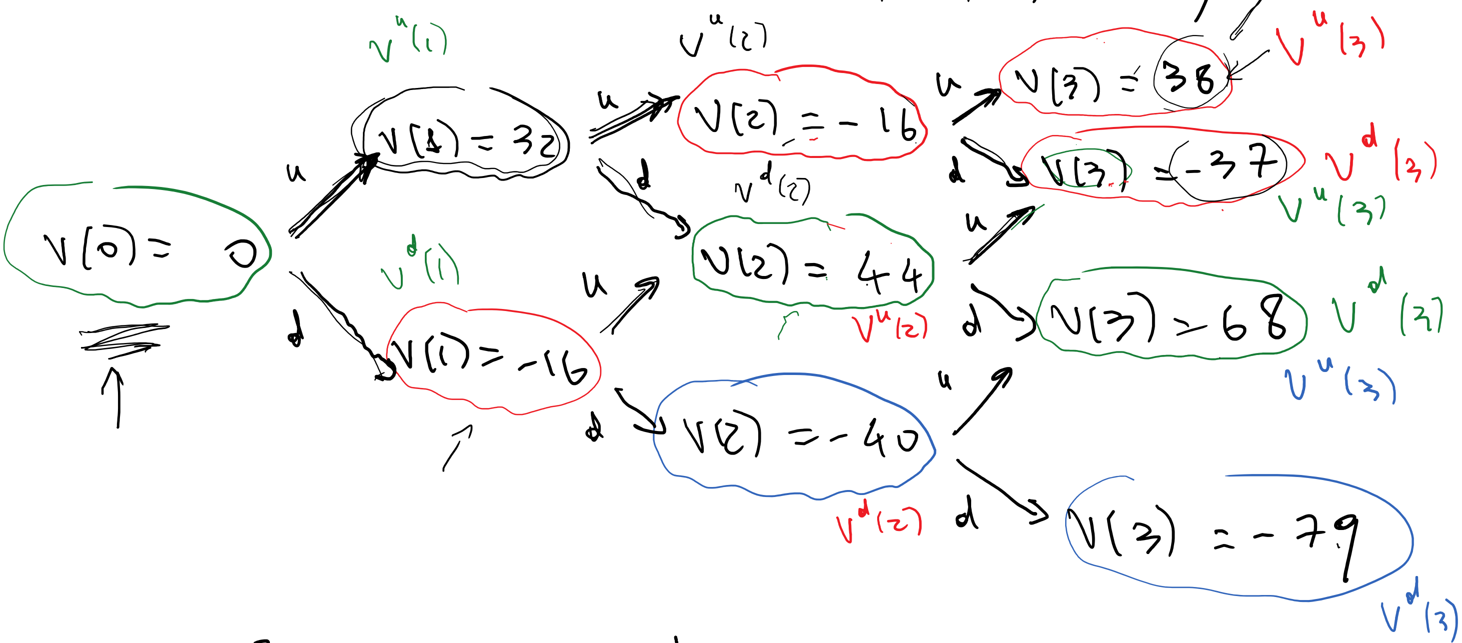
$$V(3) = h_S(3)S(3) + h_B(3)B(3) \begin{cases} \to V(3,u,u,u) = 38 \\ \to V(3,u,u,d) = V(3,u,d,u) = V(3,d,u,u) \end{cases}$$

$V(3) = \alpha_S(3) S(3) + \alpha_B(3) B(3)$

$\to V(3,u,u,d) = V(3,u,d,u) = V(3,d,u,u) = -37$

$\to V(3,u,d,d) = V(3,d,u,d) = V(3,d,d,u) = 68$

$\to V(3,d,d,d) = -79$



$$V(t) = e^{-r}\left[q_u V^u(t+1) + q_d V^d(t+1)\right] \qquad t = 0, 1, \dots N-1$$

$$q_u = \frac{e^r - e^d}{e^u - e^d} \qquad q_d = 1 - q_u \qquad u = \log\frac{5}{4} \quad d = \log\frac{1}{2}$$

$$r = \log\frac{3}{4} \qquad N = 3$$

$$q_u = \frac{\frac{3}{4} - \frac{1}{2}}{\frac{5}{4} - \frac{1}{2}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3} \qquad q_d = \frac{2}{3} \qquad t = 0, 1, 2$$

WE START FROM THE END

$t = 2$ $\quad V(2) = \frac{4}{3}\left[\frac{1}{3} V^u(3) + \frac{2}{3} V^d(3)\right]$

WHEN $V(2) = V(2,u,u) = -16$ THIS FORMULA BECOMES

$$-16 = \frac{4}{3}\left[\frac{1}{3}\cdot 38 + \frac{2}{3}(-37)\right] = \frac{4}{9}\left[38 - 74\right]$$

$$-16 = \frac{4}{3}\left[\frac{1}{3}\cdot 38 + \frac{2}{3}(-37)\right] = \frac{4}{9}\left[38-74\right] = \frac{4}{9}(-36) = -16 \checkmark$$

WHEN $V(2) = V(2,u,d) = V(2,d,u) = 44$
WE OBTAIN

$$44 = \frac{4}{3}\left[\frac{1}{3}\cdot(-37) + \frac{2}{3}\cdot 68\right] = \frac{4}{9}\left[-37+136\right] = \frac{4}{9}\,99 = 44 \checkmark$$

$\underline{t=1}$

$$V(1) = \frac{4}{3}\left[\frac{1}{3}V^u(2) + \frac{2}{3}V^d(2)\right]$$

WHEN $V(1) = V(1,u) = 32$ THIS FORMULA BECOMES

$$32 = \frac{4}{3}\left[\frac{1}{3}(-16) + \frac{2}{3}44\right] = \frac{4}{9}\left[-16+88\right] = \frac{4}{9}\,72 = 32 \checkmark$$

$\underline{t=0}$

$$\underline{V(0)} = \frac{4}{3}\left[\frac{1}{3}V^u(1) + \frac{2}{3}V^d(1)\right] = \frac{4}{3}\left[\frac{1}{3}(32) + \frac{2}{3}(-16)\right] = 0$$

FORMULA FOR $V(0)$ IN THEOREM 2.5:

$$V(0) = e^{-rN} \sum_{x \in \{u,d\}^N} (q_u)^{N_u(x)} (q_d)^{N_d(x)} V(N,x)$$

IN THIS CASE IT BECOMES

$$V(0) = e^{-3r} \sum_{x \in \{u,d\}^3} (q_u)^{N_u(x)} (q_d)^{N_d(x)} V(3,x)$$

$$= \left(\frac{4}{3}\right)^3 \sum_{x \in \{u,d\}^3} \left(\frac{1}{3}\right)^{N_u(x)} \left(\frac{2}{3}\right)^{N_d(x)} \underline{V(3,x)}$$

4 POSSIBLE VALUES

$x \in \{u, d\}$

$$= \left(\frac{4}{3}\right)^3 \left[38\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^0 + (-37)\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^1 \cdot 3 \right.$$

$$\left. + 68\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^2 \cdot 3 - 79\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^3 \right] = \ldots = 0$$