Lecture_10

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Options and Mathematics: Lecture 10

Δ - HEDGING ≡ HEDGE THE DERIVATIVE BY INVESTING ON THE UNDERLYING ASSET (AND THE RISK-FREE ASSET)

November 18, 2020

Δ - HEDGING IS NOT POSSIBLE IF THE UNDERLYING IS NOT A TRADABLE ASSET. EXAMPLE: A CAPLET IS AN OPTION ON THE INTEREST RATE (SEE ATTACHED PAGE)

Replicating portfolio of European derivatives

Next we treat the important problem of building a self-financing replicating portfolio process for European derivatives.

Theorem 3.3

Consider the European derivative with pay-off Y at the time of maturity $T = N$. Then the portfolio process given by

$\Pi_Y(t)$

$$S(t) = \begin{cases} S(t-1)e^u \\ S(t-1)e^d \end{cases}$$

$$\boxed{h_S(t) = \frac{1}{S(t-1)} \frac{\Pi_Y^u(t) - \Pi_Y^d(t)}{e^u - e^d}} \quad t \in \mathcal{I}$$

$$\boxed{h_B(t) = \frac{e^{-r}}{B(t-1)} \frac{e^u \Pi_Y^d(t) - e^d \Pi_Y^u(t)}{e^u - e^d}} \quad t \in \mathcal{I}$$

, is a self-financing replicating portfolio process.

$V(t) = \Pi_Y(t)$ FOR ALL $t = 0, 1, \dots N$,

AND SO IN PARTICULAR

$V(N) = \Pi_Y(N) = Y$ (HEDGING)

Proof. We first show that the given portfolio replicates the derivative. We have

$$V(t) = h_S(t)S(t)+h_B(t)B(t) = \frac{S(t)}{S(t-1)} \overbrace{\frac{\Pi_Y^u(t) - \Pi_Y^d(t)}{e^u - e^d}}^{h_S(t)} + \frac{e^{-r}B(t)}{B(t-1)} \overbrace{\frac{e^u\Pi_Y^d(t) - e^d\Pi_Y^u(t)}{e^u - e^d}}^{h_B(t)}.$$

Note that $e^{-r}B(t)/B(t-1) = 1$, while $S(t)/S(t-1)$ is either e^u or e^d.

By straightforward calculations we obtain

SIMPLE ALGEBRA

$$\begin{aligned} V^u(t) &= h_S(t)S(t-1)e^u + h_B(t)B(t) \\ &= e^u \frac{\Pi_Y^u(t) - \Pi_Y^d(t)}{e^u - e^d} + \frac{e^u\Pi_Y^d(t) - e^d\Pi_Y^u(t)}{e^u - e^d} = \Pi_Y^u(t), \end{aligned}$$

and similarly $V^d(t) = \Pi_Y^d(t)$. Thus $V(t) = \Pi_Y(t)$, for all $t \in \mathcal{I}$, i.e., the portfolio process is replicating, and therefore also hedging, the derivative.

$$\overbrace{h_S(t-1)S(t-1) + h_B(t-1)B(t-1)}^{V(t-1)}$$
$$= h_S(t)S(t-1) + h_B(t)B(t-1)$$

As to the self-financing property, we have

SIMPLE ALGEBRA

$$\begin{aligned} h_S(t)S(t-1) + h_B(t)B(t-1) &= \frac{\Pi_Y^u(t)(1 - e^{d-r}) + \Pi_Y^d(t)(e^{u-r} - 1)}{e^u - e^d} \\ &= e^{-r}(q_u\Pi_Y^u(t) + q_d\Pi_Y^d(t)) = \Pi_Y(t-1), \end{aligned}$$

REPLACE t WITH $t-1$ IN THE RECURRENCE FORMULA

where we used the definition of q_u, q_d, as well as the recurrence formula for $\Pi_Y(t)$.

By the already proven fact that $V(t) = \Pi_Y(t)$, for all $t \in \mathcal{I}$, we have

$$h_S(t)S(t-1) + h_B(t)B(t-1) = V(t-1), = \Pi_Y(t-1)$$

which proves the self-financing property.

$$h_S(t) = \frac{1}{S(t-1)} \frac{\Pi^u_Y(t) - \Pi^d_Y(t)}{e^u - e^d} \equiv F(S(t-1))$$

AND SIMILARLY FOR $h_B(t)$

Finally we show that the portfolio is predictable. Assume first that the European derivative is standard, i.e., $Y = g(S(N))$. Then $\Pi_Y(t) = v_t(S(t))$, and therefore

$v_t(S(t-1)e^u)$

$v_t(S(t-1)e^d)$

$$\Pi^u_Y(t) = v_t(S(t-1)e^u), \quad \Pi^d_Y(t) = v_t(S(t-1)e^d),$$

i.e., $\Pi^u_Y(t)$ and $\Pi^d_Y(t)$ are deterministic functions of $S(t-1)$. It follows that $h_S(t), h_B(t)$ are also deterministic functions of $S(t-1)$, and so this portfolio process is predictable.

In the case of non-standard derivatives we have similarly that $\Pi^u_Y(t)$ and $\Pi^d_Y(t)$ are deterministic functions of $S(1), \dots, S(t-1)$, which again implies that the portfolio is predictable. □

Example.

Let us compute the position on the stock in the hedging portfolio for the example of standard derivative considered before.

When the stock price goes up in the first step we have $S(1) = S(1,u) = 2$ and $\Pi^u_Y(2) = \Pi^u_Y(2,u) = \Pi_Y(2,u,u) = 1$, $\Pi^d_Y(2) = \Pi^d_Y(2,u) = \Pi_Y(2,u,d) = \sqrt{2} - 1$, hence

$\frac{1}{2}(2-\sqrt{2}) = 1 - \frac{\sqrt{2}}{2}$

$$h_S(2,u) = \frac{1}{S(1,u)} \frac{\Pi^u_Y(2,u) - \Pi^d_Y(2,u)}{e^u - e^d} = \frac{1}{2}\frac{1-(\sqrt{2}-1)}{2-1} = 2\cancel{-\sqrt{2}} > 0 \quad \text{(long position)}.$$

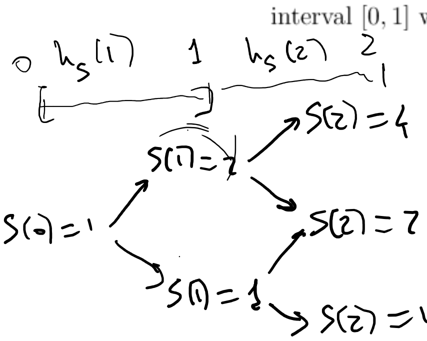
When $S(1) = S(1,d) = 1$ we have $\Pi^u_Y(2) = \Pi^u_Y(2,d) = \Pi_Y(2,d,u) = \sqrt{2} - 1$ and $\Pi^d_Y(2) = \Pi^d_Y(2,d) = \Pi_Y(2,d,d) = 0$,

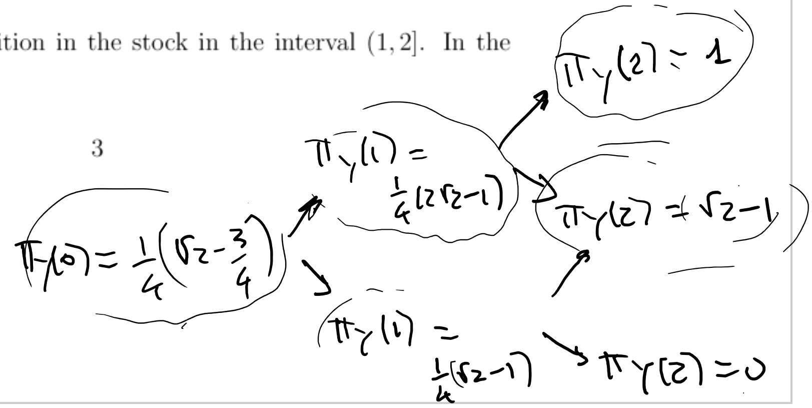
hence

CORRECT

$$h_S(2,d) = \frac{1}{S(1,d)} \frac{\overbrace{\Pi^u_Y(2,d)}^{\sqrt{2}-1} - \overbrace{\Pi^d_Y(2,d)}^{0}}{e^u - e^d} = \sqrt{2} - 1 > 0 \quad \text{(long position)}.$$

Recall that $h_S(2)$ is the position in the stock in the interval $(1,2]$. In the interval $[0,1]$ we have





$$h_S(t) = \frac{1}{S(t-1)} \frac{\Pi^u_Y(t) - \Pi^d_Y(t)}{e^u - e^d}$$

$$\frac{1}{2}\sqrt{2} - \cancel{\frac{1}{4}} - \frac{1}{4}\sqrt{2} + \cancel{\frac{1}{4}}$$

REPLACE $t=1$

$$h_S(1) = \frac{1}{S(0)}\frac{\Pi^u_Y(1) - \Pi^d_Y(1)}{e^u - e^d} = \frac{1}{\cancel{2}\ 1}\frac{\frac{1}{4}(2\sqrt{2}-1) - \frac{1}{4}(\sqrt{2}-1)}{2-1} = \frac{\sqrt{2}}{4} > 0 \quad \text{(long position).}$$

CORRECT

$$\approx \frac{1.41}{4} \approx 0.35$$

The result can be expressed in a binomial tree as follows:

$$\begin{tikzcd}
 & h_S(2) = 2 - \sqrt{2} \\
h_S(1) = \frac{\sqrt{2}}{4} \arrow[ur, "u"] \arrow[dr, "d"'] & \\
 & h_S(2) = \sqrt{2} - 1
\end{tikzcd}$$

The position on the risk-free asset can be computed likewise using the formula for $h_B(t)$ in the theorem.

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Exercise 3.11

Consider a 3-period binomial market with the following parameters:

$$e^u = \frac{5}{4}, \quad e^d = \frac{1}{2}, \quad e^r = 1 \quad p = \frac{1}{2}.$$

Assume $S_0 = \frac{64}{25}$.

Consider the European derivative expiring at time $T = 3$ and with pay-off

$$Y = S(3)H(S(3) - 1),$$

where H is the Heaviside function (this is an example of **physically-settled digital option**).

Compute the possible paths of the derivative price and for each of them give the number of shares of the underlying stock in the self-financing hedging portfolio process.

Solution to Exercise 3.11

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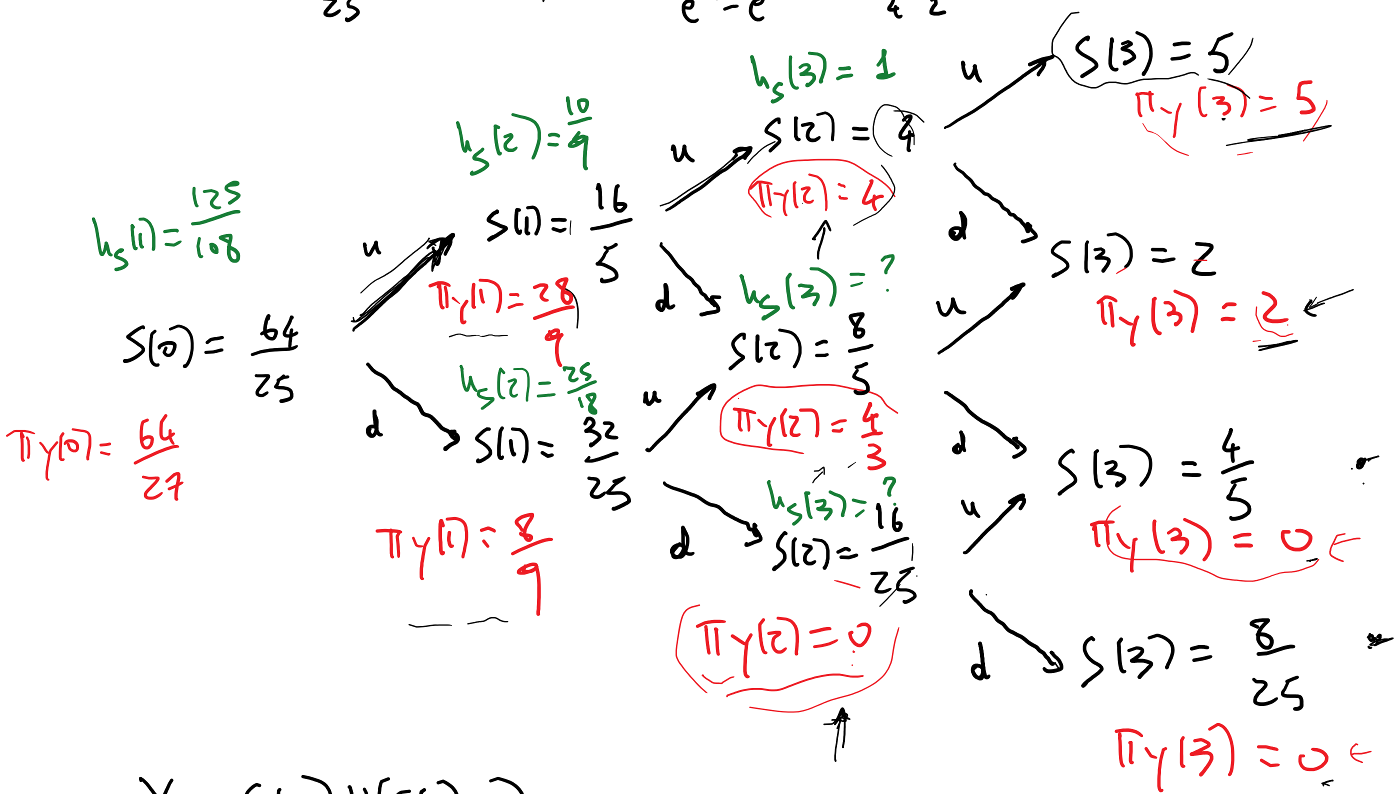
$$e^u = \frac{5}{4}, \quad e^d = \frac{1}{2} \quad e^r = 1 \quad p = \frac{1}{2}$$

$$S_0 = \frac{64}{25} \quad N=3, \quad q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{1-\frac{1}{2}}{\frac{5}{4}-\frac{1}{2}} = \frac{1/2}{3/4} = \frac{2}{3}, \quad q_d = \frac{1}{3}$$



$$Y = S(3)\,H(S(3)-1)$$

$$\Pi_Y(t) = e^{-r}\left(q_u \Pi_Y^u(t+1) + q_d \Pi_Y^d(t+1)\right)$$

$$= \frac{2}{3}\Pi_Y^u(t+1) + \frac{1}{3}\Pi_Y^d(t+1)$$

$$S(2) = 4 \Rightarrow \Pi_Y(2) = \frac{2}{3}\cdot 5 + \frac{1}{3}\cdot 2 = \frac{10}{3} + \frac{2}{3} = 4$$

$$S(1) = \frac{16}{5} \Rightarrow \Pi_Y(1) = \frac{2}{3}\cdot 4 + \frac{1}{3}\frac{4}{3} = \frac{24+4}{9} = \frac{28}{9}$$

$$\Pi_Y(0) = \frac{2}{3}\cdot\frac{28}{9} + \frac{1}{3}\cdot\frac{8}{9} = \frac{1}{27}(56+8) = \frac{64}{27}$$

SELF-FINANCING REPLICATING PORTFOLIO

$$h_S(t) = \frac{1}{S(t-1)} \frac{\Pi_Y^u(t) - \Pi_Y^d(t)}{e^u - e^d} \qquad t = 1, 2, 3$$

$h_S(1)$ (POSITION ON THE STOCK IN THE INTERVAL $[0,1]$)

$$= \frac{1}{S(0)} \frac{\Pi_Y^u(1) - \Pi_Y^d(1)}{3/4} = \frac{25}{64} \cdot \frac{4}{3} \left(\Pi_Y^u(1) - \Pi_Y^d(1) \right)$$

$$= \frac{25}{16} \frac{1}{3} \left(\frac{28}{9} - \frac{8}{9} \right) = \frac{25}{16} \cdot \frac{1}{3} \cdot \frac{20}{9} = \frac{125}{108}$$

$$= h_S(0)$$

$h_S(2, u)$ (POSITION ON THE STOCK IN THE INTERVAL $(1, 2]$ ASSUMING THE STOCK PRICE GOES UP AT $t=1$)

$$= \frac{1}{S(1,u)} \frac{\Pi_Y^u(2) - \Pi_Y^d(2)}{e^u - e^d} = \frac{1}{S(1,u)} \frac{\Pi_Y(2,u,u) - \Pi_Y(2,u,d)}{e^u - e^d}$$

$$= \frac{1}{\frac{16}{5}} \frac{1}{3/4} \left(4 - \frac{4}{3} \right) = \frac{10}{9}$$

$h_S(3, u, u)$ (POSITION ON THE STOCK IN THE INTERVAL $[2, 3]$ ASSUMING THAT THE STOCK PRICE GOES UP IN THE FIRST AND THE SECOND STEP)

$$= \frac{1}{S(2,u,u)} \frac{\Pi_Y^u(3) - \Pi_Y^d(3)}{e^u - e^d} = \frac{1}{S(2,u,u)} \frac{\Pi_Y(3,u,u,u) - \Pi_Y(3,u,u,d)}{e^u - e^d}$$

$$= \frac{1}{4} \frac{1}{3/4} (5-2) = 1$$

caplet

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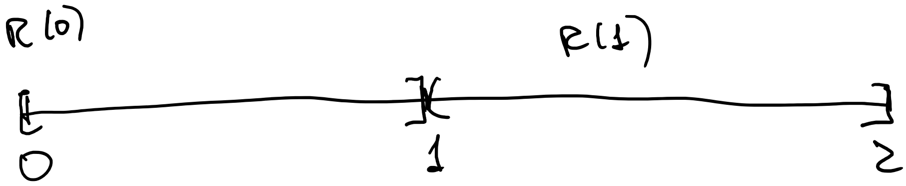

$R(0) \equiv$ INTEREST RATE IN THE TIME PERIOD $[0,1]$
IT IS KNOWN AT TIME $t=0$

$R(1) \equiv$ INTEREST RATE IN THE PERIOD $[1,2]$ AND
IS NOT KNOWN AT $t=0$.

(CAP)

CAPLET WITH MATURITY $T=1$ AND STRIKE K
IS THE CALL OPTION WITH PAY-OFF

$$Y = (R(1) - K)_+$$