Lecture_11

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Options and Mathematics: Lecture 11

November 19, 2020

Exercises

$0 \qquad T_1 = 1 \qquad T_2 = 2$

Exercise 3.3

A **compound option** is an option whose underlying is another option.

For instance, given $T_2 > T_1 > 0$ and $K_1, K_2 > 0$, a **call on a put** with maturity T_1 and strike K_1 is a contract that gives to its owner the right to buy at time T_1 for the price K_1 the put option on the stock with maturity T_2 and strike K_2.

Let $S(t)$ be the price of the underlying stock of the put option. Assume that $S(t)$ follows a 2-period binomial model with parameters

$$e^u = \frac{7}{4}, \quad e^d = \frac{1}{2}, \quad e^r = \frac{9}{8}, \quad p = \frac{1}{4}, \quad S(0) = 16.$$

Assume further that $T_2 = 2$, $T_1 = 1$, $K_1 = \frac{23}{9}$, $K_2 = 12$.

Compute the initial price of the call on the put. Compute also the probability of positive return for the owner of the call on the put.

IN THE INTERVAL $[0, 2]$

Exercise 3.23

Compute the self-financing hedging portfolio for the compound option in the previous exercise. Assume $B(0) = 1$.

Exercise 3.6

A **barrier option** is an option that expires worthless as soon as the stock price crosses a specified level (the barrier of the option

For example, consider a 3-period binomial market with parameters

$$e^u = \frac{4}{3}, \quad e^d = \frac{2}{3}, \quad p = \frac{3}{4}, \quad S_0 = 2, \quad andr = 0$$

and the barrier call option with strike $K = 2$ and barrier $B = 3$.

This means that the derivative expires worthless if $S(3) \leq 2$ or if $S(t) > 3$ at some time $t \in \{1, 2, 3\}$.

Compute the binomial price $\Pi_Y(0)$ of this barrier option at time $t = 0$ and the probability that it expires in the money.

ANSWER: $\Pi_Y(0) = 5/54$. $\mathbb{P}(Y > 0) = 28,125\%$.

Exercise 3.7

Let $N = 2$ and

$$e^u = \frac{5}{4}, \quad e^d = \frac{1}{2}, \quad e^r = 1, \quad S_0 = \frac{64}{25}, \quad p \in (0,1).$$

Consider a European derivative with maturity $T = 2$ and pay-off

$$Y = \Big(\frac{1}{3}\sum_{i=0}^{2} S(i) - 2\Big)_+,$$

which is an example of **Asian** call option. Compute the price of the derivative at times $t = 0, 1, 2$.

ANSWER: $\Pi_Y(0) = 148/225$, $\Pi_Y(1,u) = 74/75$, $\Pi_Y(2,u,u) = 94/75$, $\Pi_Y(2,u,d) = 34/75$, $\Pi_Y(1,d) = \Pi_Y(2,d,u) - \Pi_Y(2,d,d) = 0$.

Exercise 3.8 (WE WILL DO IT LATER IN THE COURSE)

The Asian call, resp. put, with strike K in a N-period binomial market is the non-standard European derivative with pay-off

$$Y_{\mathrm{AC}}(x) = \left(\frac{1}{N+1}\sum_{t=0}^{N} S(t) - K\right)_+, \quad \text{resp.} \quad Y_{\mathrm{AP}}(x) = \left(K - \frac{1}{N+1}\sum_{t=0}^{N} S(t)\right)_+.$$

Derive the following put-call parity satisfied by the binomial price at time $t = 0$ of the Asian option:

$$\Pi_{\mathrm{AC}}(0) - \Pi_{\mathrm{AP}}(0) = e^{-rN}\Big[\frac{S(0)}{N+1}\frac{e^{r(N+1)} - 1}{e^r - 1} - K\Big].$$

HINT: For $\alpha \neq 1$, $\sum_{k=0}^{N} \alpha^k = \frac{1-\alpha^{N+1}}{1-\alpha}$.

Solution of Exercise 3.3

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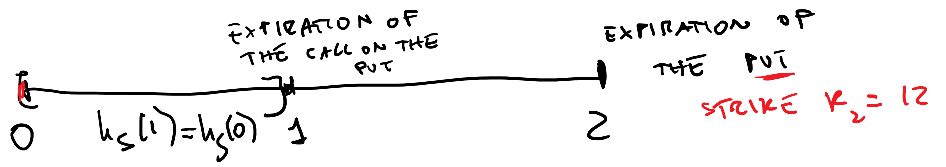
PAY OF OF THE CALL ON THE PUT ?



$$\boxed{Y_{\substack{\text{CALL}\\\text{ON PUT}}} =} \left(\Pi_{\text{PUT}}(1) - K_1\right)_+ \qquad Y_{\text{PUT}} = (12 - S(2))_+$$

$$\Pi_{\text{put}}(t) = v_t(S(t))$$

$$= \left(\Pi_{\text{put}}(1) - \frac{23}{9}\right)_+ \Leftarrow \Big(\underbrace{v_1(S(1))}_{\Pi_{\text{put}}(1)} - \frac{23}{9} \Big)_+$$

$$= g(S(1))$$

$S_0 = 16$, $\Pi_{\text{put}}(0) = \frac{128}{81}$

$\xrightarrow{u}$ $S(1) = 28$, $\Pi_{\text{put}}(1) = 0$
- $\xrightarrow{u}$ $S(2) = 49$, $\Pi_{\text{put}}(2) = 0$
- $\xrightarrow{d}$ $S(2) = 14$, $\Pi_{\text{put}}(2) = 0$

$\xrightarrow{d}$ $S(1) = 8$, $\Pi_{\text{put}}(1) = \frac{32}{9}$
- $\xrightarrow{u}$ $S(2) = 14$
- $\xrightarrow{d}$ $S(2) = 4$, $\Pi_{\text{put}}(2) = 8$

$$\Pi_{\text{put}}(1) = e^{-r}\left(q_u \cancel{\Pi^u_{\text{put}}(2)} + q_d \Pi^d_{\text{put}}(2)\right) =$$

$$= \frac{8}{9}\left(\frac{1}{2} \cdot 8\right) = \frac{32}{9}$$

$$q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{9/8 - 1/2}{7/4 - 1/2} = \frac{5/8}{5/4} = \frac{1}{2} = q_d$$

$$Y_{\substack{\text{CALL}\\\text{ON PUT}}} = \begin{cases} 0 & \text{IF STOCK GOES UP AT TIME 1} \\ \frac{32}{9} - \frac{23}{9} = 1 & \text{IF STOCK GOES DOWN AT TIME 1} \end{cases}$$

$$\xrightarrow{u} \Pi_{\substack{\text{CALL}\\\text{ON PUT}}}(1) = 0 = \Pi^u_{\substack{\text{CALL}\\\text{ON PUT}}}(1)$$

$$\Pi_{\substack{CALL\\ON PUT}}(0) = \begin{cases} \xrightarrow{u} \Pi_{\substack{CALL\\ON PUT}}(1) = 0 = \Pi^u_{\substack{CALL\\ON PUT}}(1) \\ \xrightarrow{d} \Pi_{\substack{CALL\\ON PUT}}(1) = 1 = \Pi^d_{\substack{CALL\\ON PUT}}(1) \end{cases}$$

$$\Pi_{\substack{CALL\\ON PUT}}(0) = e^{-r}\left(q_u \cancel{\Pi^u_{\substack{CALL\\ON PUT}}(1)}^{\,0} + q_d \underbrace{\Pi^d_{\substack{CALL\\ON PUT}}(1)}_{=1}\right)$$

$$= \frac{8}{9}\,\frac{1}{2}\cdot 1 = \frac{4}{9}$$

PROBABILITY OF POSITIVE RETURN

$$R(u,u) = 0 - \frac{4}{9} = -\frac{4}{9}$$

$$R(u,d) = 0 - \frac{4}{9} = -4/9$$

$$R(d,u) = 0 - \frac{4}{9} - \frac{23}{9} = -\frac{27}{9} = -3$$

$$R(d,d) = 8 - \frac{4}{9} - \frac{23}{9} = 8 - 3 = 5$$

$$\mathbb{P}(R > 0) = \mathbb{P}(S^{(d,d)}) = (1-p)^2 = \left(\frac{3}{4}\right)^2 \approx 56{,}25\%$$

EXERCISE 3.23 (HEDGING PORTFOLIO FOR THIS COMPOUND OPTION)

$$h_S(1) = \frac{1}{S(0)} \frac{\Pi^u_{\substack{CALL\\ON PUT}}(1) - \Pi^d_{\substack{CALL\\ON PUT}}(1)}{e^u - e^d} = \frac{1}{16} \frac{(-1)}{7/4 - \frac{1}{2}} =$$

$$= - \frac{4/5}{16} = \underline{-\frac{1}{20}}$$

CAREFUL: IT IS $\boxed{\underline{\text{NOT}}}$ TRUE THAT

$$\Pi_{PUT}(1) = \begin{cases} \Pi_{put}(0)e^u \\ \Pi_{put}(0)e^d \end{cases}$$

SO WE CANNOT USE THE FORMULAS IN THEOREM 3.3 WITH $S(t) = \Pi_{put}(t)$

Solution to Exercise 3.6

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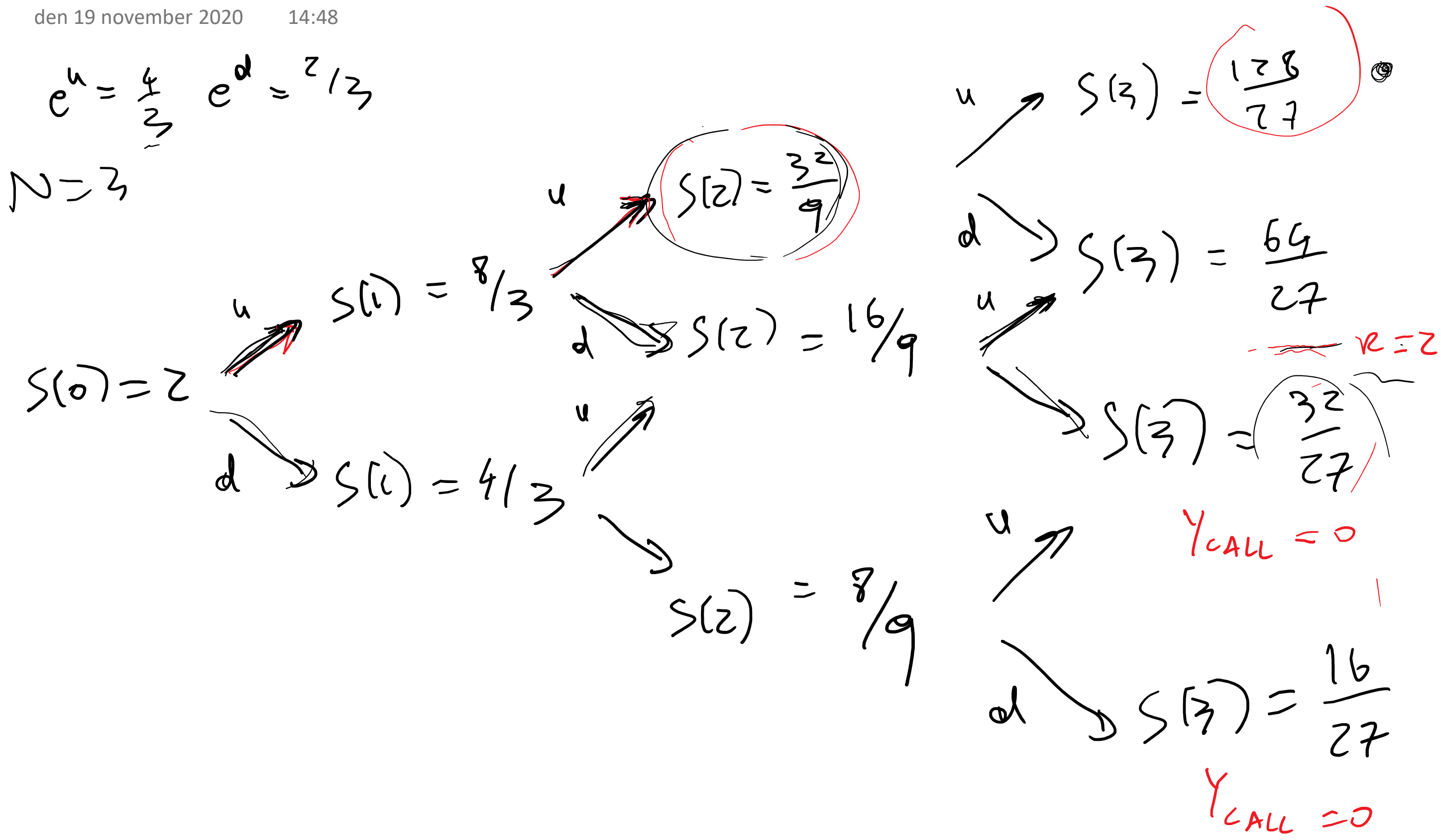

PAY-OFF ~~AT~~ MATURITY $T=3$ OF THE BARRIER OPTION $= (S(3)-2)_+$

PROVIDED THE STOCK DOES NOT EXCEED THE BARRIER $B = 3$ AT SOME TIME.

BECAUSE ALONG THESE PATHS THE STOCK PRICE CROSSES THE BARRIER

$$Y(u,u,u) = 0 \qquad Y(u,u,d) = 0$$

$$Y(u,d,u) = \frac{64}{27} - 2 = \frac{10}{27} = Y(d,u,u)$$

$$Y(d,u,d) = Y(u,d,d) = Y(d,d,u) = 0$$

$$Y(d,d,d) = 0$$

Y > 0 ONLY ON 2 PATHS : (u, d, u) AND (d, u, u)

$$\mathbb{P}(Y>0) = \mathbb{P}(S^{(u,d,u)}) + \mathbb{P}(S^{(d,u,u)}) = 2p^2(1-p)$$

$$= 2\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right) \approx 28{,}125\,\%$$

$$\Pi_Y(0) = e^{-rN} \sum_{x \in \{u,d\}^N} (q_u)^{N_u(x)} (q_d)^{N_d(x)} \underline{Y(x)}$$

$$q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{1 - 2/3}{4/3 - 2/3} = \frac{1/3}{2/3} = \frac{1}{2} = q_d$$

$$\Rightarrow \Pi_Y(0) = \left(\frac{1}{2}\right)^{3 = N} \left(Y(u,d,u) + Y(d,u,u)\right)$$

$r = 0$

$$= \frac{1}{8}\,\frac{10}{27}\cdot 2 = \frac{5}{54}$$

Solution to Exercise 3.7

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$$e^u = 5/4 \qquad e^d = \frac{1}{2} \qquad r = 0 \qquad S(0) = \frac{64}{25} \qquad N = 2$$

$$\begin{tikzcd}[column sep=large]
 & & S(2) = 4 \\
 & S(1) = \frac{16}{5} \arrow[ur, "u"] \arrow[r, "d"'] & S(2) = 8/5 \\
S(0) = \frac{64}{25} \arrow[ur, "u"] \arrow[dr, "d"'] & & \\
 & S(1) = \frac{32}{25} \arrow[uur, "u"] \arrow[r, "d"'] & S(2) = \frac{16}{25}
\end{tikzcd}$$

$$\Pi_Y(0) = e^{-2r} \sum_{x \in \{u,d\}^2} (q_u)^{N_u(x)} (q_d)^{N_d(x)} \underbrace{\left(\frac{1}{3} \sum_{i=0}^{2} S(i) - 2 \right)_+}_{Y}$$

$$q_u = \frac{1 - 1/2}{5/4 - \frac{1}{2}} = \frac{1/2}{3/4} = 2/3 \qquad q_d = \frac{1}{3}$$

$$Y(u,u) = \left(\frac{1}{3} \left(\frac{64}{25} + \frac{16}{5} + 4 \right) - 2 \right)_+$$

$$= \left(\frac{1}{3} \; \frac{64+80+100}{25} - 2 \right)_+ = \left(\frac{1}{3} \frac{244}{25} - 2 \right)_+$$

$$= \frac{244-150}{75} = \frac{94}{75}$$

$$Y(u,d) = \left(\frac{1}{3} \left(\frac{64}{25} + \frac{16}{5} + \frac{8}{5} \right) - 2 \right)$$

$$\left(\dots \, 3 \left(\dots 25 \dots 5 \dots 5 \right) - 2 \right)_+$$

$$= \left(\frac{1}{3} \left(\frac{64+80+40}{25} \right) - 2 \right)_+ = \left(\frac{184}{75} - 2 \right)_+ = \frac{34}{75}$$

$$Y(d,u) = \left(\frac{1}{3} \left(\frac{64}{25} + \frac{32}{25} + \frac{8}{5} \right) - 2 \right)_+$$

$$Y(d,d) = \left(\frac{1}{3} \underbrace{\left(\frac{64}{25} + \frac{32}{25} + \frac{16}{25} \right)} - 2 \right)_+$$

$$\Pi_Y(0) = 1 \Big((q_u)^2 (q_d)^0 \, Y(u,u) + (q_u)^1 (q_d)^1 \, Y(u,d)$$

$$+ q_u q_d \, Y(d,u) + (q_d)^2 \, Y(d,d) \Big)$$

$$= \text{----} = \frac{148}{225}$$