Exam for the course "Options and Mathematics" (CTH[*MVE095*], GU[*MMG810*]) 2020/21

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April 9$^{\text{th}}$, 2021 (8.30-12.30)

REMARKS: (1) All aids permitted, however you must work alone (2) Give all details and explain all steps in your solutions (3) Write as clear as possible: if some step is not clearly readable it will assumed to be wrong! (4) See the course homepage for instructions on how to submit the exam.

Part I

1. Prove that under suitable conditions on the market parameters, the one-period binomial market model is arbitrage-free (max 2 points). **Solution:** See Theorem 2.4 in the lecture notes.

2. Give and explain the definition of risk-neutral price at time zero of the ZCB (max 2 points). **Solution:** See Definition 6.10 in the lecture notes,

3. Assume that the market is arbitrage-free and $r = 0$. Consider an American style Lookback option on a non-dividend paying stock with maturity T and intrinsic value

$$Y(t) = (K - \min_{[0,t]} S(\tau))_+$$

where $K > 0$. Decide whether the following statements are true or false and explain your answer (max 2 points):

(a) It is never optimal to exercise this derivative prior to maturity.

(b) If $K > S(0)$ the value of the derivative tends to zero as $T \to \infty$.

Solution: (a) is true. In fact since $\min_{[0,t_2]} S(\tau) \leq \min_{[0,t_1]} S(\tau)$ for $t_2 > t_1$, then $Y(T) > Y(t)$, for all $t < T$ and since $r = 0$, then there is no loss in money value to wait until maturity to receive the largest possible pay-off. (b) is false, because $Y(T) > Y(0) = (K - S(0))_+ > 0$ and so, again because $r = 0$, the value of the derivative is also larger than $Y(0)$ for all $T > 0$, and thus cannot tend to zero when $T \to \infty$.

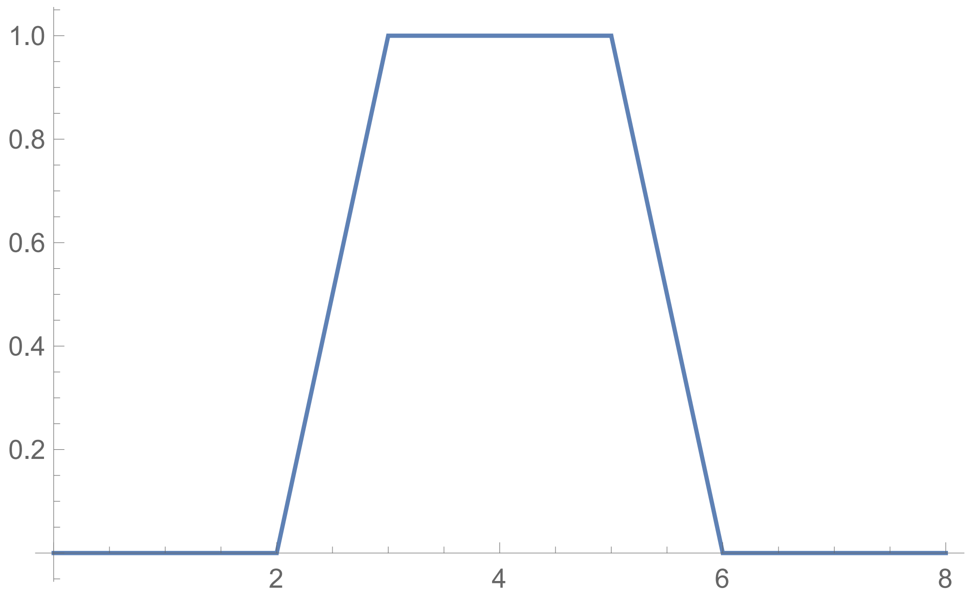
Part II

1. Consider the European style derivative with pay-off

$$Y = \min((S(T) - 2))_+, (6 - S(T))_+, H(S(T) - L)),$$

where H denotes the Heaviside function. Compute the range of values of L for which this derivative can be replicated by a portfolio on European calls and/or put options (max 2 points) and derive such portfolio (max 2 points).

Solution: A derivative can be replicated by call/put options if and only if the pay-off is piecewise linear and continuous. By drawing the graph of Y as a function of $S(T)$, one can see that this happens in two cases: when $L \geq 6$, in which case $Y \equiv 0$, and when $0 < L \leq 2$, in which case the pay-off looks like in the following picture.



Thus for $0 \leq L \leq 2$ a replicating portfolio is $(C(2), -C(3), -C(5), C(6))$, where $C(K)$ denotes the call with strike K and maturity T. For $L \geq 6$, the derivative is replicated by a portfolio on call and put options with zero value. This can only happens in the trivial case when the investor has a long and short position on the same call or put.

2. Consider a 3 period binomial model with the following parameters:

$$S(0) = 8, \quad u = \log 2, \quad d = -\log 2, \quad r = \log(5/4), \quad p = 1/3$$

and an American digital option with intrinsic value $Y(t) = H(S(t) - 10)$, where H is the Heaviside function. Compute the value of the derivative at all times (max 2 points) and the cash that the writer can withdraw from the hedging portfolio if the derivative is not exercised at the optimal exercise times prior to maturity (max 2 points).

Solution: The binomial tree of the stock price is given by

$$
\begin{tikzcd}
 & & & S(3)=64 \\
 & & S(2)=32 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & S(1)=16 \arrow[ur, "u"] \arrow[dr, "d"] & & S(3)=16 \\
S(0)=8 \arrow[ur, "u"] \arrow[dr, "d"] & & S(2)=8 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & S(1)=4 \arrow[ur, "u"] \arrow[dr, "d"] & & S(3)=4 \\
 & & S(2)=2 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & & S(3)=1
\end{tikzcd}
$$

and the binomial tree of the intrinsic value of the derivative is given by

$$
\begin{tikzcd}
 & & & Y(3)=1 \\
 & & Y(2)=1 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & Y(1)=1 \arrow[ur, "u"] \arrow[dr, "d"] & & Y(3)=1 \\
Y(0)=0 \arrow[ur, "u"] \arrow[dr, "d"] & & Y(2)=0 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & Y(1)=0 \arrow[ur, "u"] \arrow[dr, "d"] & & Y(3)=0 \\
 & & Y(2)=0 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & & Y(3)=0
\end{tikzcd}
$$

The binomial price of the American derivative is computed using the recurrence formula

$$\widehat{\Pi}_Y(3) = Y(3), \quad \widehat{\Pi}_Y(t) = \max(Y(t), e^{-r}(q_u \widehat{\Pi}_Y^u(t+1) + q_d \widehat{\Pi}_Y^d(t+1))), \quad t = 0, 1, 2,$$

where $q_u = \frac{e^r - e^d}{e^u - e^d} = 1/2 = q_d$. It follows that the binomial tree for $\widehat{\Pi}_Y(t)$ is

$$
\begin{tikzcd}
 & & & \widehat{\Pi}_Y(3) = 1 \\
 & & \boxed{\widehat{\Pi}_Y(2) = 1} \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & \boxed{\widehat{\Pi}_Y(1) = 1} \arrow[ur, "u"] \arrow[dr, "d"] & & \widehat{\Pi}_Y(3) = 1 \\
\widehat{\Pi}_Y(0) = 58/125 \arrow[ur, "u"] \arrow[dr, "d"] & & \widehat{\Pi}_Y(2) = 2/5 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & \widehat{\Pi}_Y(1) = 4/25 \arrow[ur, "u"] \arrow[dr, "d"] & & \widehat{\Pi}_Y(3) = 0 \\
 & & \widehat{\Pi}_Y(2) = 0 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & & \widehat{\Pi}_Y(3) = 0
\end{tikzcd}
$$

The optimal exercise times prior to maturity are those within a box. At time 1, when $S(1) = 16$, the cash that can be withdrawn by the replicating portfolio, if the buyer does not exercise, is

$$C(1) = Y(1) - e^{-r}(q_u \widehat{\Pi}_Y^u(2) + q_d \widehat{\Pi}_Y^d(2)) = 1 - \frac{4}{5}\left(\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{2}{5}\right) = \frac{11}{25}$$

and similarly one finds $C(2) = 1/5$ when $S(2) = 32$.

3. Find the risk-neutral price at time zero of the European style derivative with maturity $T > 0$ and pay-off

$$Y = H\left(e^{\frac{1}{T}\int_0^T \log S(t)\, dt} - K\right)$$

where $K > 0$ and H denotes the Heaviside function (max 4 points).

Solution: In the risk-neutral probability we have

$$\log S(t) = \log S(0) + (r - \sigma^2/2)t + \sigma \widetilde{W}(t).$$

Replacing in the pay-off we obtain

$$Y = H\left(S(0)e^{(r-\sigma^2/2)T/2} e^{\sigma \int_0^T \widetilde{W}(t)\, dt} - K\right)$$

Since $\int_0^T W(t)\, dt \in \mathcal{N}(0, T^3/3)$, we obtain

$$Y = H\left(S(0)e^{(r-\sigma^2/2)T/2} e^{\sigma\sqrt{\frac{3}{T^3}}G} - K\right)$$

where $G \in \mathcal{N}(0,1)$ in the risk-neutral probability. By the risk-neutral pricing formula the price $\Pi_Y(0)$ of the derivative is

$$\Pi_Y(t) = e^{-rT}\widetilde{\mathbb{E}}[Y] = e^{-rT}\int_{\mathbb{R}} H\left(S(0)e^{(r-\sigma^2/2)T/2}e^{\sigma\sqrt{\frac{3}{T^3}}x} - K\right)e^{-x^2/2}\frac{dx}{\sqrt{2\pi}}.$$

Computing the integral we find

$$\Pi_Y(0) = e^{-rT}\Phi(-d), \quad \text{where} \quad d = \frac{\log\frac{K}{S(0)} - \left(r - \frac{\sigma^2}{2}\right)\frac{T}{2}}{\sigma\sqrt{\frac{3}{T^3}}}.$$