Lecture_13

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Options and Mathematics: Lecture 13

November 24, 2020

Replicating portfolios of American derivatives

$Y(t) = g(S(t))$

Definition 4.3

A predictable portfolio process $\{h_S(t), h_B(t)\}_{t\in\mathcal{I}}$ is said to be hedging the American derivative with intrinsic value $Y(t)$ and maturity $T = N$ if

$$V(N) = Y(N), \quad V(t) \geq Y(t),\ t = 0, \dots, N-1,$$

where $V(t) = h_S(t)S(t) + h_B(t)B(t)$ is the value of the portfolio process at time t.

If $V(t) = \widehat{\Pi}_Y(t)$ for all $t = 0, \dots, N$, the portfolio process is said to be **replicating** the American derivative.

IF $V(t) = \hat{\Pi}_Y(t)$, THEN SINCE $\hat{\Pi}_Y(t) \geq Y(t)$, WE ALSO HAVE $V(t) \geq Y(t)$

Replicating portfolio processes are hedging portfolios, because $\widehat{\Pi}_Y(t) \geq Y(t)$.

In the European case any self-financing hedging portfolio process is (trivially) replicating, because $\Pi_Y(t)$ has been defined as the common value of any such portfolio.

[?] WHY DON'T WE REQUIRE THAT $V(t) = Y(t)$ INSTEAD OF $V(t) \geq Y(t)$?

IN THE AMERICAN CASE ONE CANNOT EXPECT THAT REPLICATING PORTFOLIOS ARE SELF-FINANCING. IN FACT IF $V(t) = \hat{\Pi}_Y(t)$ AND THE BUYER EXERCISES AT A NON-OPTIMAL EXERCISE TIME t, THAT IS $\hat{\Pi}_Y(t) > Y(t)$, THEN THE SELLER PORTFOLIO CONTAINS MORE VALUE THAN REQUIRED TO PAY-OFF THE BUYER AND SO THE SELLER CAN WITHDRAW THE CASH $\hat{\Pi}_Y(t) - Y(t)$

However, in the American case, the class of self-financing portfolios is in general too small to contain replicating portfolio processes.

The following theorem shows that replicating portfolios for American derivatives may generate a cash flow.

Theorem 4.3

Consider the standard American derivative with intrinsic value $Y(t)$ and maturity $T = N$.

$= g(S(t))$

Let $\widehat{\Pi}_Y(t)$ be its binomial price of the American derivative.

Define the portfolio process $\{(\widehat{h}_S(t), \widehat{h}_B(t))\}_{t\in\mathcal{I}}$ and the cash flow process $C(t)$ as follows: For $t = 1, \ldots, N-1$,

$$\boxed{C(0) = 0, \quad C(t) = \widehat{\Pi}_Y(t) - e^{-r}[q_u \widehat{\Pi}^u_Y(t+1) + q_d \widehat{\Pi}^d_Y(t+1)]}$$

$\to \max(Y(t), e^{-r}(q_u \hat{\Pi}^u_Y(t+1) + q_d \hat{\Pi}^d_Y(t+1)))$

and, for $t = 1, \ldots, N$,

$$\boxed{\widehat{h}_S(t) = \frac{1}{S(t-1)} \frac{\widehat{\Pi}^u_Y(t) - \widehat{\Pi}^d_Y(t)}{e^u - e^d}}$$

$$\boxed{\widehat{h}_B(t) = \frac{e^{-r}}{B(t-1)} \frac{e^u \widehat{\Pi}^d_Y(t) - e^d \widehat{\Pi}^u_Y(t)}{e^u - e^d}}$$

ARE EXACTLY THE SAME AS IN THE EUROPEAN CASE

This portfolio process is predictable, replicates the American derivative and generates the cash-flow $C(t)$

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By the previous theorem the writer can hedge the derivative and still be able to withdraw cash from the portfolio.

Whether the writer is allowed or not to withdraw cash from the portfolio (i.e., $C(t) > 0$) depends on the "smartness" of the buyer.

In fact, we have, for $t \in \{1, \dots, N-1\}$,

$$C(t) = \underbrace{\max(Y(t), e^{-r}(q_u\widehat{\Pi}^u_Y(t+1)) + q_d\widehat{\Pi}^d_Y(t+1))}_{\Pi_Y(t)} - e^{-r}(q_u\widehat{\Pi}^u_Y(t+1)) + q_d\widehat{\Pi}^d_Y(t+1)).$$

This quantity is positive at time t if and only if only

$$\boxed{Y(t) > e^{-r}(q_u\widehat{\Pi}^u_Y(t+1)) + q_d\widehat{\Pi}^d_Y(t+1))}$$

(C(t) = 0 if t is not an optimal exercise time

which implies that t is an optimal exercise time.

Hence the writer of the American put can withdraw cash from the portfolio only if the buyer fails to exercise the derivative optimally.

If however the buyer exercises the derivative optimally, then the seller needs the full value of the portfolio to pay-off the buyer and thus no cash can be withdrawn.

IF THIS HAPPENS, THEN

$$\hat{\Pi}_Y(t) = \max(Y(t), e^{-r}(q_u\Pi^u_Y(t+1) + q_d\Pi^d_Y(t+1)))$$

$$= Y(t),$$ i.e., t is an optimal exercise time

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Exercise 4.4

Consider the American derivative with intrinsic value

$$Y(t) = \min(S(t), (24 - S(t))_+)$$

and expiring at time $T = 3$. The initial price of the underlying stock is $S(0) = 27$, while at future times it follows the binomial model

$e^u = 4/3 \quad e^d = 2/3$

$p = 1/2$

$r = 0$

$$S(t+1) = \begin{cases} 4S(t)/3 & \text{with probability } 1/2 \\ 2S(t)/3 & \text{with probability } 1/2 \end{cases}$$

for $t = 0, 1, 2$. Assume also that the risk-free rate of the money market is zero. Compute the possible paths of the binomial price of the derivative. In which case it is optimal for the buyer to exercise the derivative prior to expiration? What is the amount of cash that the seller can withdraw from the portfolio if the buyer does not exercise the derivative optimally?

Exercise 4.5 (DO IT YOURSELF)

Consider a 3-period binomial model with the following parameters:

$$e^u = \frac{5}{4}, \quad e^d = \frac{1}{2}, \quad e^r = 1, \quad p \in (0, 1).$$

Let $S(0) = \frac{64}{25}$ be the initial price of the stock. Consider an American style derivative on the stock with maturity $T = 3$ and intrinsic value

$$Y(t) = |3 - S(t)|\, H\big(S(t) - 7/5\big),$$

where $H(x)$ is the Heaviside function and $|x|$ is the absolute value of x. Compute the binomial price of the derivative at each time $t \in \{0, 1, 2, 3\}$ and the initial position on the stock in the replicating portfolio. Compute the cash that the seller can withdraw from the portfolio if the buyer does not exercise the derivative at optimal times. Compute the probability that the derivative is in the money at time $t \in \{0, 1, 2, 3\}$ and the probability that the return for the buyer is positive at time $t \in \{0, 1, 2, 3\}$ if the derivative is exercised at this time.

Exercise 4.12

Assume $e^u = \frac{7}{4}$, $e^d = \frac{1}{2}$, $S(0) = 1$, $p = 3/4$, $e^r = 9/8$.

a) Compute the binomial price at $t = 0, 1, 2$ of an American put with strike $K = 3/4$ and maturity $T = 2$

b) Compute the binomial price at $t = 0, 1, 2$ of a European call with strike $K = 3/4$ and maturity $T = 2$

c) A derivative $\mathcal{U}$ gives to its owner the right to convert $\mathcal{U}$ at time $t = 1$ into either the European call or the American put defined above. Compute the binomial price of $\mathcal{U}$ at time $t = 0$

d) Describe the strategy that the holder of $\mathcal{U}$ should follow.

ANSWER: c) $\Pi_{\mathcal{U}}(0) = 16/27$.

Solution to Exercise 4.4

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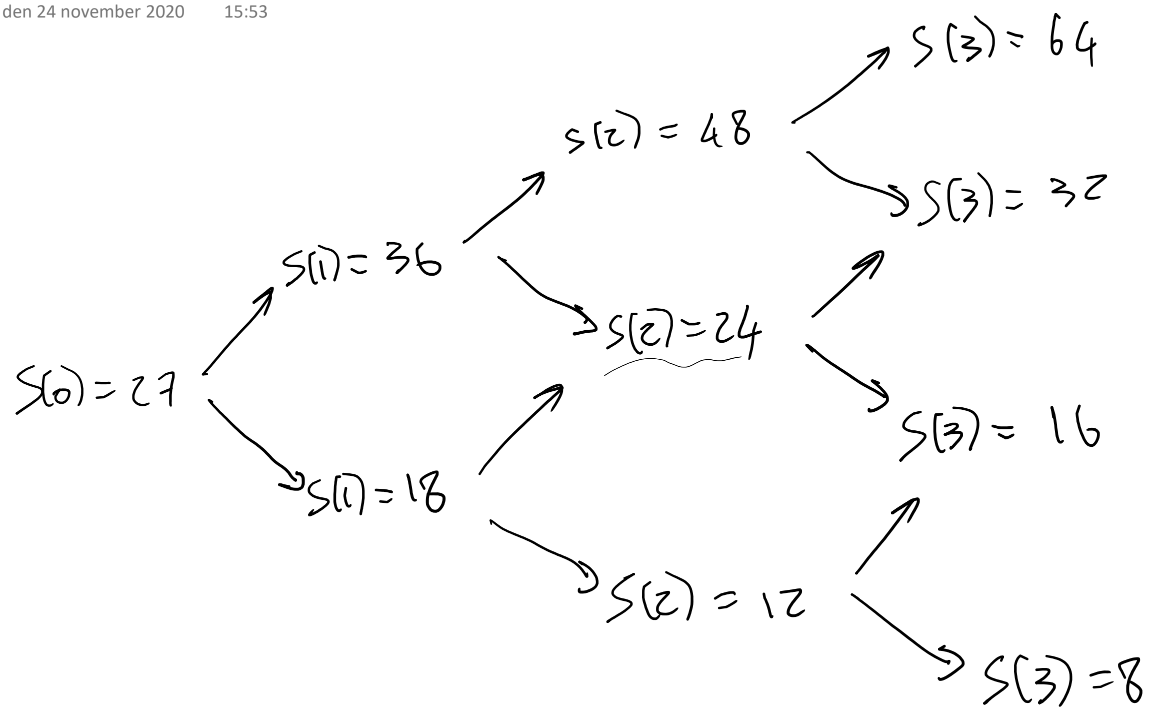
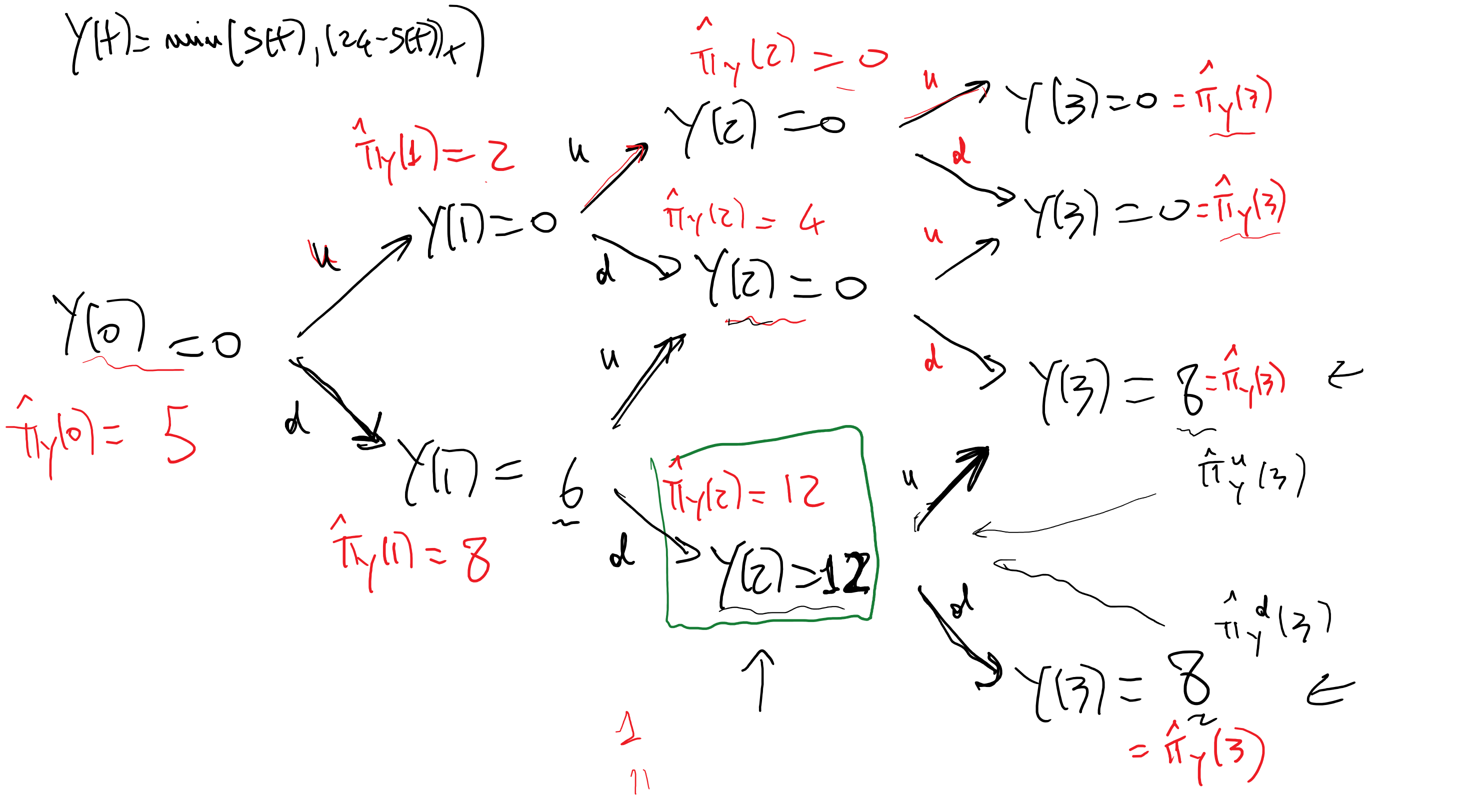


$$Y(t) = \min\left(S(t), (24 - S(t))_+\right)$$



$$\hat{\pi}_Y(t) = \max\left(Y(t),\ \overset{1}{\overset{\|}{e^{-r}}}\left(q_u \hat{\pi}_Y^u(t+1) + q_d \hat{\pi}_Y^d(t+1)\right)\right)$$

$$q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{1 - 2/3}{4/3 - 2/3} = \frac{1/3}{2/3} = \frac{1}{2} = q_d$$

IF $x_1 = d$, $x_2 = d$ $\quad \hat{\pi}_Y(2) = \max\left(12, \frac{1}{2}(8+8)\right) = 12$

IF $x_1 = d$ $\hat{\Pi}_Y(1) = \max\left(6, \frac{1}{2}(4+12)\right) = 8$

ONLY WHEN THE STOCK PRICE GOES DOWN IN THE FIRST 2 STEPS IT IS OPTIMAL TO EXERCISE THE DERIVATIVE PRIOR TO MATURITY

IF $x_1 = x_2 = d$ AND THE BUYER DOES NOT EXERCISE THE DERIVATIVE, THEN THE SELL CAN WITHDRAW

$$C(2) = \hat{\Pi}_Y(2) - e^{-r}\left(q_u \hat{\Pi}_Y^u(3) + q_d \hat{\Pi}_Y^d(3)\right)$$

$$= 12 - \frac{1}{2}(8+8) = 12 - 8 = \underline{4}$$

Solution to Exercise 4.12

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$e^u = 7/4 \qquad e^d = \frac{1}{2} \qquad e^r = 9/8 \qquad S(0) = 1 \qquad p = 3/4$

$\boxed{N=2}$

$S(0) = 1$

- $S(1) = 7/4$, $\quad \Pi_Y(1) = \frac{4}{9}\left(\frac{37}{16} + \frac{1}{8}\right) = \frac{4}{9}\frac{39}{16} = \frac{13}{12}$
  - $S(2) = 49/16$, $\quad \Pi_Y(2) = \left(\frac{49}{16} - \frac{3}{4}\right)_+ = \frac{37}{16}$
  - $S(2) = 7/8$, $\quad \Pi_Y(2) = \left(\frac{7}{8} - \frac{3}{4}\right)_+ = \frac{1}{8}$
- $S(1) = 1/2$, $\quad \Pi_Y(1) = \frac{4}{9}\frac{1}{8} = \frac{1}{18}$
  - $S(2) = 7/8$
  - $S(2) = \frac{1}{4}$, $\quad \Pi_Y(2) = \left(\frac{1}{4} - \frac{3}{4}\right)_+ = 0$

$\Pi_Y(0) = \frac{4}{9}\left(\frac{13}{12} + \frac{1}{18}\right) = \frac{4}{9}\left(\frac{39+2}{36}\right) = \frac{41}{81}$

$Y(t) = \left(\frac{3}{4} - S(t)\right)_+$

$Y(0) = 0$, $\quad \hat{\Pi}_Y(0) = 1/9$

- u: $Y(1) = 0$, $\quad \hat{\Pi}_Y(1) = 0$
  - u: $Y(2) = 0 = \hat{\Pi}_Y(2)$
  - d: $Y(2) = 0 = \hat{\Pi}_Y(2)$
- d: $Y(1) = 1/4$, $\quad \hat{\Pi}_Y(1) = 1/4$
  - u
  - d: $Y(2) = 1/2 = \hat{\Pi}_Y(2)$

OPTIMAL EXERCISE

$$\hat{\Pi}_Y(t) = \max\left(Y(t),\ e^{-r}\left(q_u \hat{\Pi}^u_Y(t+1) + q_d \hat{\Pi}^d_Y(t+1)\right)\right) \qquad t = 0, 1$$

$$q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{9/8 - 1/2}{7/4 - 1/2} = \frac{5/8}{5/4} = 1/2 = q_d$$

IF $x_1 = d$, $\hat{\Pi}_Y(1) = \max\left(\frac{1}{4}, \frac{8}{9}\left(\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot\frac{1}{2}\right)\right)$

$$= \max\left(\frac{1}{4}, \frac{2}{9}\right) = 1/4$$

$$\hat{\Pi}_Y(0) = \frac{8}{9}\cdot\frac{1}{2}\cdot\frac{1}{4} = 1/9$$

FOR THE EUROPEAN CALL WITH $K = 3/4$ WE USE THE FORMULA

$$\Pi_Y(t) = e^{-r}\left(q_u \Pi_Y^u(t+1) + q_d \Pi_Y^d(t+1)\right)$$

$$= \frac{8}{9}\left(\frac{1}{2}\Pi_Y^u(t+1) + \frac{1}{2}\Pi_Y^d(t+1)\right) = \frac{4}{9}\left(\Pi_Y^u(t+1) + \Pi_Y^d(t+1)\right)$$

$t = 0, 1$

$$\Pi_Y(2) = Y = \left(S(2) - 3/4\right)_+$$

THE PAY-OFF OF U IS $\max\left(\hat{\Pi}_Y(1), \Pi_Y(1)\right)$

$\hat{\Pi}_Y(1)$: American put with strike 3/4

$\Pi_Y(1)$: European call with strike 3/4

$\Pi_U(0) = \frac{16}{27}$

u: $\max\left(0, \frac{13}{12}\right) = \frac{13}{12}$

d: $\max\left(\frac{1}{4}, \frac{1}{18}\right) = \frac{1}{4}$

$$\Pi_u(0) = e^{-r}\left(q_u \Pi_Y^u(1) + q_d \Pi_Y^d(1)\right) = \frac{8}{9}\cdot\frac{1}{2}\left(\frac{13}{12}+\frac{1}{4}\right)$$

$$= \frac{4}{9}\ \frac{13+3}{12} = \frac{4\cdot 16}{9\cdot 12_3} = \frac{16}{27}$$

$$\Pi_Y(t) = e^{-r}\left(q_u \Pi_Y^u(t+1) + q_d \Pi_Y^d(t+1)\right)$$

$$\Pi_Y(0) = e^{-rN}\sum_{t\in\{u,d\}^N}(q_u)^{N_u(t)}(q_d)^{N_d(t)}\,Y(t)$$