Lecture_16

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Options and Mathematics: Lecture 16

November 27, 2020

Review of finite probability theory

Expectation and Variance

We may think of the expectation of X as an estimate on the average value of X and the variance of X as a measure of how far is this estimate from to the precise value of X.

Definition 5.9

Let $(\Omega, \mathbb{P})$ be a finite probability space and $X : \Omega \to \mathbb{R}$ a random variable. The **expectation** (or **expected value**) of X is defined by

$$\mathbb{E}[X] = \sum_{i=1}^{M} X(\omega_i)\mathbb{P}(\omega_i).$$

We shall write the definition of $\mathbb{E}[X]$ also as

NOTATION $p_X(x)$

$$\boxed{\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega)\mathbb{P}(\{\omega\})} = \sum_{x \in \mathrm{Im}(X)} x \, \mathbb{P}(X = x)$$

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$\omega \in \Omega \Rightarrow X(\omega) \in \mathrm{Im}(X)$

$x \in \mathrm{Im}(X)$

$\Omega_N = \{H, T\}^N = \{\omega = (\gamma_1, \gamma_2, \dots \gamma_N)$ WHERE $\gamma_i = H$ OR $T\}$

$p \in (0,1)$ $\quad \mathbb{P}_p(\{\omega\}) = p^{N_H(\omega)} (1-p)^{N_T(\omega)}$ $\quad N_T(\omega) = N - N_H(\omega)$

P ≡ PROBABILITY TO GET A HEAD IN EACH TOSS

Example In the N-coin toss probability space $(\Omega_N, \mathbb{P}_p)$ we have

$$\mathbb{E}_p[X] = \sum_{\omega \in \Omega_N} X(\omega) p^{N_H(\omega)} (1-p)^{N_T(\omega)},$$

where $N_H(\omega)$ is the number of heads and $N_T(\omega) = N - N_H(\omega)$ is the number of tails in the N-toss $\omega \in \Omega_N$.

We can rewrite the definition of expectation as

$$\mathbb{E}[X] = \sum_{x \in \mathrm{Im}(X)} x \, \mathbb{P}(X = x),$$

or equivalently

$$\boxed{\mathbb{E}[X] = \sum_{x \in \mathrm{Im}(X)} x f_X(x)}$$

The importance of the previous formula is that it allows to compute the expectation of X from its distribution, without any reference to the original probability space.

Example

If we are told that a random variable X takes the following values:

$\mathrm{Im}(X) = \{1, 2, -1\}$

$$X = \begin{cases} 1 & \text{with probability } 1/4 \\ 2 & \text{with probability } 1/4, \\ -1 & \text{with probability } 1/2 \end{cases}$$

$X(\omega)$

REALIZATION OF THE R.V. X

then we can compute $\mathbb{E}[X]$ as

$$\mathbb{E}[X] = 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} - 1 \cdot \frac{1}{2} = \frac{1}{4}.$$

$\mathrm{VAR}[X] = \sum_{x \in \mathrm{Im}(X)} x^2 f_X(x)$ 2 $- \mathbb{E}[X]^2 = 1 \cdot \frac{1}{4} + 4 \frac{1}{4} + (-1)^2 \frac{1}{2} - \left(\frac{1}{4}\right)^2 = \frac{24}{16}$

Some simple properties of the expectation are collected in the following theorem.

Theorem 5.2

Let X, Y be random variables on a finite probability space $(\Omega, \mathbb{P})$, $g : \mathbb{R} \to \mathbb{R}$, $a, b \in \mathbb{R}$. The following holds:

1. $\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$ (linearity).
2. If $X \geq 0$ and $\mathbb{E}[X] = 0$, then $X = 0$.
3. If X, Y are independent, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.
4. If $Y = g(X)$, i.e., if Y is X-measurable, then

PROOF IN THE BOOK

$$\mathbb{E}[g(X)] = \mathbb{E}[Y] = \sum_{x \in \mathrm{Im}(X)} g(x) f_X(x). \tag{1}$$

$g(z) = z^2$

Definition 5.10

Let $(\Omega, \mathbb{P})$ be a finite probability space. The **variance** of a random variable $X : \Omega \to \mathbb{R}$ is defined by

$$\mathrm{Var}[X] = \mathbb{E}[\underbrace{(\mathbb{E}[X] - X)^2}_{\geq 0}].$$

BY 2 OF THEOREM 5.2, VAR[X] = 0 IFF X = E[X], i.e., IFF X IS A DETERMINISTIC CONSTANT

Using the linearity of the expectation, we obtain easily the formula

$$\boxed{\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2}$$

Remarks

- The variance of a random variable is always non-negative and it is zero if and only if the random variable is a deterministic constant. Hence we may also interpret the variance as a measure of the "randomness" of a random variable.
- $\mathrm{Var}[aX] = a^2\mathrm{Var}[X]$ holds for all constants $a \in \mathbb{R}$, and

  $$\mathrm{Var}[X+Y] = \mathbb{E}[(X+Y)^2] - \mathbb{E}[X+Y]^2 = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]).$$

  THIS IS ZERO IF X, Y ARE INDEPENDENT.

  It follows by Theorem ~~3~~ 5.2(3) that the variance of the sum of two independent random variables is the sum of their variance

Using ~~(3)~~ (4) in Theorem 5.2 with $g(x) = x^2$, we can rewrite the definition of variance in terms of the distribution function of X as

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$\mathbb{E}[g(X)], \quad g(z) = z^2$$

$$\mathrm{Var}[X] = \sum_{x\in\mathrm{Im}(X)} x^2 f_X(x) - \left(\sum_{x\in\mathrm{Im}(X)} x f_X(x)\right)^2,$$

which allows to compute $\mathrm{Var}[X]$ without any reference to the original probability space.

For instance for the random variable on page 2 we find

$$\mathrm{Var}[X] = 1 \cdot \frac{1}{4} + 4 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} - \left(\frac{1}{4}\right)^2 = \frac{27}{16}.$$

$$f_X(x) = \begin{cases} 0 & \text{IF } x \neq u,\ x \neq d \\ p & \text{IF } x = u \\ 1-p & \text{IF } x = d \end{cases}$$

h

$t_0 = 0 \quad t_1 \quad t_2 \quad t_{i-1} \quad t_i \quad \cdots \quad T = t_N$

$$S(t_{i-1}) \qquad S(t_i) = \begin{cases} S(t_{i-1})\, e^u \\ S(t_{i-1})\, e^d \end{cases}$$

$\alpha \qquad \sigma$

Example: mean of log return and volatility of the binomial stock price

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of the interval $[0, T]$ with $t_i - t_{i-1} = h$, for all $i = 1, \ldots, N$.

Given $u > d$ and $p \in (0, 1)$, consider a random variable X such that $X = u$ with probability p and $X = d$ with probability $1 - p$.

A POSSIBLE REALIZATION OF X

We may think of X as being defined on $\Omega_1 = \{H, T\}$ (ω_1, ω_2), with $X(H) = u$ (ω_1) and $X(T) = d$ (ω_2).

The binomial stock price at time t_i can be written as $S(t_i) = S(t_{i-1}) \exp(X)$.

Hence the log-return R of the stock in the interval $[t_{i-1}, t_i]$ is

$$R = \log S(t_i) - \log S(t_{i-1}) = \log \frac{S(t_i)}{S(t_{i-1})} = X.$$

It follows that the expectation and the variance of the log-return of the stock in the interval $[t_{i-1}, t_i]$ are

$$= \sum_{x \in Im(X)} x \cdot f_X(x)$$

$$\mathbb{E}[R] = \mathbb{E}[X] = (pu + (1-p)d),$$

$$\mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$\mathrm{Var}[R] = \mathrm{Var}[X] = [pu^2 + (1-p)d^2 - (pu + (1-p)d)^2)] = p(1-p)(u-d)^2.$$

Thus the parameters α, σ^2 in the binomial model can be rewritten as

$$\boxed{\alpha = \frac{1}{h}\mathbb{E}[R], \quad \sigma^2 = \frac{1}{h}\mathrm{Var}[R]}$$

$$\alpha = \frac{pu + (1-p)d}{h}$$

$$\sigma^2 = \frac{p(1-p)(u-d)^2}{h}$$

It is part of our assumptions on the binomial model that the parameters α and σ are the same for every interval $[t_{i-1}, t_i]$ of the partition.

IF $Y = g(X)$, Y CARRIES THE SAME INFORMATION AS X

X, Y ARE INDEPENDENT, THEN THEY CARRY INDEPENDENT INFORMATION

Conditional expectation

If X, Y are independent random variables, knowing the value of Y does not help to estimate the random variable X.

However if X, Y are not independent, then we can use the information carried by Y to find an estimate of X which is better than $\mathbb{E}[X]$. This leads to the important concept of **conditional expectation**.

Definition 5.14

Let $(\Omega, \mathbb{P})$ be a finite probability space, $X, Y : \Omega \to \mathbb{R}$ random variables and $y \in \mathrm{Im}(Y)$. The expectation of X conditional to $Y = y$ (or given the event $\{Y = y\}$) is defined as

$$\mathbb{E}[X] = \sum_{x \in \mathrm{Im}(X)} \mathbb{P}(X = x)\, x$$

$$\boxed{\mathbb{E}[X|Y = y] = \sum_{x \in \mathrm{Im(X)}} \mathbb{P}(X = x|Y = y)\, x}$$

where $\mathbb{P}(X = x|Y = y)$ is the conditional probability of the event $\{X = x\}$, given the event $\{Y = y\}$.

$$\mathbb{P}(X = x | Y = y) = \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(Y = y)}$$

The random variable

$$\mathbb{E}[X|Y] : \Omega \to \mathbb{R}, \quad \mathbb{E}[X|Y](\omega) = \mathbb{E}[X|Y = Y(\omega)] \qquad \omega \in \Omega$$

is called the expectation of X conditional to Y.

In a similar fashion one defines the conditional expectation with respect to several random variables, i.e., $\mathbb{E}[X|Y_1 = y_1, Y_2 = y_2, \ldots, Y_N = y_N]$ and $\mathbb{E}[X|Y_1, \ldots, Y_N]$.

$\omega_1 \; \omega_2 \; \omega_3 \; \omega_4 \; \omega_5 \; \omega_6$

$\Omega = \{1, 2, 3, 4, 5, 6\}$

Example

$\mathbb{P}(\{\omega_i\}) = 1/6$

In the probability space of a fair die, consider

$\operatorname{Im}(X) = \{-1, 1\}$

$$X(\omega) = (-1)^{\omega}, \quad Y(\omega) = (\omega - 1)(\omega - 2)(\omega - 3), \quad \omega \in \{1, 2, 3, 4, 5, 6\}.$$

Note that $\operatorname{Im}(Y) = \{0, 6, 24, 60\}$. Then we compute

$\mathbb{E}[X \mid Y = y]$

$y \in \{0, 6, 24, 60\}$

$\sum_{x \in \operatorname{Im}(X)} \mathbb{P}(X = x \mid Y = 0)\, x$

$$\begin{aligned}
\mathbb{E}[X|Y=0] &= \mathbb{P}(X=1|Y=0) - \mathbb{P}(X=-1|Y=0) \\
&= \frac{\mathbb{P}(X=1, Y=0)}{\mathbb{P}(Y=0)} - \frac{\mathbb{P}(X=-1, Y=0)}{\mathbb{P}(Y=0)} \\
&= \frac{\mathbb{P}(\{2\})}{\mathbb{P}(\{1,2,3\})} - \frac{\mathbb{P}(\{1,3\})}{\mathbb{P}(\{1,2,3\})} = -1/3.
\end{aligned}$$

$\dfrac{1/6}{1/2} - \dfrac{1/3}{1/2}$

Similarly we find

$$\mathbb{E}[X|Y=6] = 1, \quad \mathbb{E}[X|Y=24] = -1, \quad \mathbb{E}[X|Y=60] = 1,$$

hence $\mathbb{E}[X|Y]$ is the random variable

$$\mathbb{E}[X \mid Y = Y(\omega)] = \mathbb{E}[X|Y](\omega) = \begin{cases} -1/3 & \text{if } \omega = 1, 2 \text{ or } 3 \\ 1 & \text{if } \omega = 4 \text{ or } 6 \\ -1 & \text{if } \omega = 5. \end{cases}$$

$Y(\omega) = 0$

$Y(4) = 6, \; Y(6) = 60$

$Y(5) = 24$

The following theorem collects a few important properties of the conditional expectation that will be used later on.

Theorem 5.3

Let $X, Y, Z : \Omega \to \mathbb{R}$ be random variables on the finite probability space $(\Omega, \mathbb{P})$. Then

OBVIOUS (0) The random variable $\mathbb{E}[X|Y]$ is Y-measurable; $\mathbb{E}[X|Y] = g(Y)$

OBVIOUS (1) The conditional expectation is a linear operator, i.e.,

$$\mathbb{E}[\alpha X + \beta Y|Z] = \alpha\mathbb{E}[X|Z] + \beta\mathbb{E}[Y|Z],$$

for all $\alpha, \beta \in \mathbb{R}$;

(2) If X is independent of Y, then $\mathbb{E}[X|Y] = \mathbb{E}[X]$; Y GIVES NO NEW INFORMATION

(3) If X is measurable with respect to Y, i.e., $X = g(Y)$ for some function g, then $\mathbb{E}[X|Y] = X$; IF I KNOW Y, THEN I KNOW X

(4) $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$; THE INFORMATION ON Y HAS BEEN LOST

$X = f(Z)$ (5) If X is measurable with respect to Z, then $\mathbb{E}[XY|Z] = X\mathbb{E}[Y|Z]$; TAKE OUT WHAT IS KNOWN

(6) If Z is measurable with respect to Y then $\mathbb{E}[\mathbb{E}[X|Y]|Z] = \mathbb{E}[X|Z]$. TOWER PROPERTY

Z CONTAINS LESS INFORMATION THAN Y

These properties remain true if the conditional expectation is taken with respect to several random variables. → I LOOSE THE LARGEST INFORMATION

Remarks

- The interpretation of (2) is the following: If X is independent of Y, then the information carried by Y does not help to improve our estimate on X and thus our best estimate for X remains $\mathbb{E}[X]$.
- The interpretation of (3) is the following: if X is measurable with respect of Y, then by knowing Y we also know X and thus our best estimate on X is X itself.

Stochastic processes

$X(t): \Omega \to \mathbb{R}$
$\omega \rightsquigarrow X(t)(\omega) \in \mathbb{R}$

Let $(\Omega, \mathbb{P})$ be a finite probability space and $T > 0$.

A one parameter family of random variables, $X(t) : \Omega \to \mathbb{R}$, $t \in [0, T]$, is called a **stochastic process**.

We denote the stochastic process by $\{X(t)\}_{t\in[0,T]}$ and by $X(t, \omega)$ the value of the random variable $X(t)$ on the sample $\omega \in \Omega$.

$X(t)(\omega) = X(t, \omega)$
(NOTATION)

For each fixed $\omega \in \Omega$, the curve $t \to X(t, \omega)$, is called a **path** of the stochastic process.

We shall refer to the parameter t as the time variable, as this is what it represents in most applications.

If $X(t, \omega) = C(t)$, for all $\omega \in \Omega$, i.e., if the paths are the same for all sample points, we say that the stochastic process is a **deterministic function** of time.

If t runs over a (possibly finite) discrete set $\{t_0, t_1, \dots\} \subset [0, T]$, then we say that the stochastic process is **discrete**.

$\{X(t_0), X(t_1), X(t_2), \dots\}$

Note that a discrete stochastic process is equivalent to a sequence of random variables:

$$\{X_0, X_1, \dots\}, \quad \text{where } X_i = X(t_i),\ i = 0, 1, \dots.$$

If the discrete stochastic process is finite, i.e., if it runs only for a finite number N of time steps, we shall denote it by $\{X_n\}_{n=0,\dots,N}$ and call it a N**-period** (STOCHASTIC) **process**. If it runs for infinitely many steps we denote it by $\{X_n\}_{n\in\mathbb{N}}$.

RECALL THAT A RANDOM VARIABLE Y IS SAID TO BE MEASURABLE W.R.T. A SECOND R.V. X IF $Y = g(X)$ FOR SOME FUNCTION $g: \mathbb{R} \to \mathbb{R}$

Definition 5.15

Let $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ be two discrete stochastic processes on a finite probability space.

The process $\{Y_n\}_{n\in\mathbb{N}}$ is said to be **measurable** with respect to $\{X_n\}_{n\in\mathbb{N}}$ if for all $n \in \mathbb{N}$ there exists a function $g_n : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $Y_n = g_n(X_0, X_1, \ldots, X_n)$.

If $Y_n = h_n(X_0, \ldots, X_{n-1})$ for some function $h_n : \mathbb{R}^n \to \mathbb{R}$, then $\{Y_n\}_{n\in\mathbb{N}}$ is said to be **predictable** from the process $\{X_n\}_{n\in\mathbb{N}}$.

Example: The random walk.

Consider the following (discrete and finite) stochastic process $\{X_n\}_{n=1,\ldots,N}$ defined on the N-coin toss probability space $(\Omega_N, \mathbb{P}_p)$:

$\omega \in \Omega_N$, $\omega = (\gamma_1, \ldots, \gamma_N)$ WHERE $\gamma_i = H$ OR T

$\Omega_N = \{H, T\}^N$

$n = 1, \ldots, N$

$$\omega = (\gamma_1, \ldots, \gamma_N) \in \Omega_N, \quad X_n(\omega) = \begin{cases} 1 & \text{if } \gamma_n = H \\ -1 & \text{if } \gamma_n = T \end{cases}$$

Clearly, the random variables $X_1, \ldots, X_N$ are independent and identically distributed (**i.i.d**), namely

$\mathbb{P}_p(\omega) = p^{N_H(\omega)} (1-p)^{N_T(\omega)}$

$\operatorname{Im}(X_n) = \{-1, 1\}$

$f_{X_n}(x) = \begin{cases} p & \text{IF } x = 1 \\ 1-p & \text{IF } x = -1 \\ 0 & \text{IF } x \neq \pm 1 \end{cases}$

$$\mathbb{P}_p(X_n = 1) = p, \quad \mathbb{P}_p(X_n = -1) = 1 - p, \quad \text{for all } n = 1, \ldots, N.$$

Hence

$\mathbb{E}[X_n] = 1 \cdot p + (-1)(1-p) = 2p - 1$

$$\mathbb{E}[X_n] = 2p - 1, \quad \text{Var}[X_n] = 4p(1-p), \quad \text{for all } n = 1, \ldots, N.$$

Now, for $n = 1, \ldots, N$, let

$\{M_n\}_{n=0, \ldots, N}$,

$$M_0 = 0, \quad M_n = \sum_{i=1}^{n} X_i.$$

The stochastic process $\{M_n\}_{n=0,\ldots,N}$ is called the (N**-period**) **random walk**.

$M_0 = 0$, $M_1 = X_1$, $M_2 = X_1 + X_2$, $M_3 = X_1 + X_2 + X_3$, ...

$$\mathbb{E}[M_n] = \mathbb{E}[X_1 + X_2 + \ldots + X_n]$$

It satisfies

$$= \mathbb{E}[X_1] + \ldots + \mathbb{E}[X_n]$$

$$\mathbb{E}[M_n] = n(2p-1), \quad \text{for all } n = 0, \ldots, N. \qquad = n(2p-1)$$

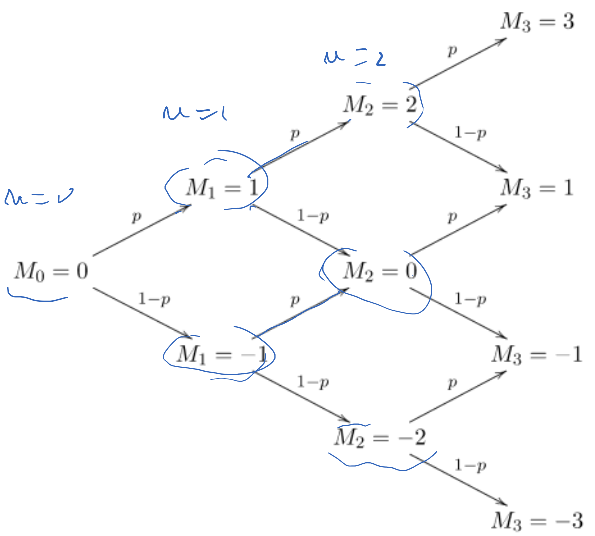
Moreover, being the sum of independent random variables, the random walk has variance given by

BECAUSE $X_1, \ldots X_n$ ARE INDEPENDENT

$$\mathrm{Var}[M_0] = 0, \quad \mathrm{Var}[M_n] = \mathrm{Var}(X_1 + X_2 + \cdots + X_n) = \sum_{i=1}^{n} \mathrm{Var}[X_i] = 4np(1-p).$$

When $p = 1/2$, the random walk is said to be **symmetric**. In this case $\{M_n\}_{n=0,\ldots,N}$ satisfies $\mathbb{E}[M_n] = 0$, $n = 0, \ldots, N$ and $\mathrm{Var}[M_n] = n$. When $p \neq 1/2$, $\{M_n\}_{n=0,\ldots,N}$ is called **asymmetric** random walk, or random walk with **drift**.

If $M_n = k$ then M_{n+1} is either $k+1$ (with probability p), or $k-1$ (with probability $1-p$). Hence we can represent the paths of the random walk by using a binomial tree, as in the following example for $N = 3$:



RECOMBINING BINOMIAL TREE

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Martingales

A martingale is a stochastic process which has no tendency to rise or fall. The precise definition is the following.

Definition 5.16

A discrete stochastic process $\{X_n\}_{n\in\mathbb{N}}$ on the finite probability space $(\Omega, \mathbb{P})$ is called a **martingale** if

$$\mathbb{E}[X_{n+1}|X_0, X_1, \ldots, X_n] = X_n, \quad \text{for all } n \in \mathbb{N}.$$

Interpretation: The variables $X_0, X_1, \ldots, X_n$ contains the information obtained by "looking" at the stochastic process up to the step n. For a martingale process, this information is not enough to estimate whether, in the next step, the process will raise or fall.

Remarks

$\mathbb{P} : 2^{\Omega_N} \to [0,1]$

$\widetilde{\mathbb{P}} : 2^{\Omega_N} \to [0,1]$

1. The martingale property depends on the probability being used: if $\{X_n\}_{n\in\mathbb{N}}$ is a martingale in the probability $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ is another probability measure on the sample space Ω, then $\{X_n\}_{n\in\mathbb{N}}$ need not be a martingale with respect to $\widetilde{\mathbb{P}}$.

2. Using property 4 in Theorem 5.3 we obtain

$$\boxed{\mathbb{E}[X_{n+1}] = \mathbb{E}[X_n],} \quad \text{for all } n \in \mathbb{N}.$$

Thus, iterating, $\mathbb{E}[X_n] = \mathbb{E}[X_0]$, for all $n \in \mathbb{N}$, i.e., *martingales have constant expectation.* !!

$$\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$$

$$\mathbb{E}[\mathbb{E}[X_{n+1}|X_0, \ldots X_n]] = \mathbb{E}[X_n]$$

$$\|$$

$$\mathbb{E}[X_{n+1}]$$

$$\mathbb{E}[X_{n+1}] = \mathbb{E}[X_n] = \mathbb{E}[X_{n-1}] = \ldots = \mathbb{E}[X_0]$$

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$\mathbb{E}[M_{n+1} | M_0, M_1, \dots M_n] = M_n$

Example

Next we show that the *symmetric* random walk is a martingale.

Using the linearity of the conditional expectation we have, for all $n = 0, \dots, N-1$,

$M_{n+1} = M_n + X_{n+1}$

$$\begin{aligned}\mathbb{E}[M_{n+1}|M_0,\dots,M_n] &= \mathbb{E}[M_n + X_{n+1}|M_0,\dots,M_n] \\ &= \underbrace{\mathbb{E}[M_n|M_0,\dots,M_n]}_{M_n} + \underbrace{\mathbb{E}[X_{n+1}|M_0,\dots,M_n]}_{=0}.\end{aligned}$$

$M_n = \sum_{i=1}^{n} X_i$

X_{n+1}

As M_n is measurable with respect to $M_0, \dots, M_n$ then

$$\mathbb{E}[M_n|M_0,\dots,M_n] = M_n$$

see Theorem 5.3(3).

Moreover, as X_{n+1} is independent of $M_0, \dots, M_n$, Theorem 5.3(2) gives

$$\mathbb{E}[X_{n+1}|M_0,\dots,M_n] = \mathbb{E}[X_{n+1}] = 0 \quad p = 1/2$$

It follows that $\mathbb{E}[M_{n+1}|M_0,\dots,M_n] = M_n$, i.e., the symmetric random walk is a martingale.

However the asymmetric random walk $(p \neq 1/2)$ is *not* a martingale, as it follows by the fact that its expectation $\mathbb{E}[M_n] = n(2p-1)$ is not constant (it depends on $n \in \mathbb{N}$).

$X(t)$ STOCHASTIC PROCESS

FIX THE SAMPLE POINT $\omega=\omega_*$

AND LET t RUNS IN $[0,T]$

IF I FIX $\omega=\omega_{**}$

I GET ANOTHER PATH