Lecture_18

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Lecture_18

Options and Mathematics: Lecture 18

December 2, 2020

Exercises

Execise 5.22 (SEE ALSO EXERCISE 3.8)

Consider a N-period binomial market with $r \neq 0$ and let $S(t)$ denote the price of the stock at time $t \in \{0, \dots, N\}$. The Asian call, resp. put, with maturity $T = N$ and strike K is the non-standard European style derivative with pay-off

$$Y_{\text{call}} = \left[\left(\frac{1}{N+1}\sum_{t=0}^{N} S(t)\right) - K\right]_+ , \quad \text{resp.} \quad Y_{\text{put}} = \left[K - \left(\frac{1}{N+1}\sum_{t=0}^{N} S(t)\right)\right]_+ .$$

Denote by $AC(0)$ and $AP(0)$ the binomial price at time $t = 0$ of the Asian call and put, respectively. Use the risk-neutral pricing formula to prove the following put-call parity identity:

$$AC(0) - AP(0) = e^{-rN}\left[\frac{S(0)}{N+1}\frac{e^{r(N+1)} - 1}{e^r - 1} - K\right].$$

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HINT: For $\alpha \neq 1$, $\sum_{k=0}^{N} \alpha^k = \frac{1 - \alpha^{N+1}}{1 - \alpha}$

PUT-CALL PARITY FOR EUROPEAN OPTIONS: $C - P = S - K e^{-r(T-t)}$

IMPORTANT! REQUIRES THE FORMULA
$\Pi_Y(t) = e^{-r(N-t)} \mathbb{E}_q[Y | S(0), \dots, S(t)]$

Execise 5.23

$\Pi_Y^*(t) = e^{-rt}\Pi_Y(t)$

Use the risk-neutral pricing formula to show the discounted binomial price $\{\Pi_Y^*(t)\}_{t=0,\dots,N}$ of European derivatives is a martingale in the risk-neutral probability measure and to give an alternative proof of the recurrence formula for the binomial price of European derivatives. $\Pi_Y(t) = e^{-r}[q\,\Pi_Y^u(t+1) + (1-q)\Pi_Y^d(t+1)]$

Next consider the augmented binomial market consisting of the stock, the risk-free asset and a European derivative on the stock priced by the risk-neutral pricing formula. Use the martingale property of $\{S^*(t)\}_{t=0,\dots,N}$ and $\{\Pi_Y^*(t)\}_{t=0,\dots,N}$ to show that this market does not admit self-financing arbitrages.

Exercise 5.32

$e^u = 5/4 \quad e^d = 1/2$

$r = 0$

Consider a 2-period binomial model with the parameters $u = \log(5/4)$, $d = \log(1/2)$, $r = 0$, $S(0) = 64/25$, $p \in (0, 1)$.

Compute the price at time $t \in \{0, 1, 2\}$ of the American put on the stock with maturity $T = 2$ and strike price $K_2 = \frac{11}{5}$ and identify the possible optimal exercise times prior to maturity.

Next consider the compound option which gives to its owner the right to buy the American put at time $t = 1$ for the price $K_1 = \frac{8}{25}$. Compute the price of the compound option at time $t = 0$ and the hedging portfolio for the compound option (assume $B(0) = 1$).

Derive the strategy that maximises the expected return for the owner of the compound option, where as usual we assume that the investor can only exercise, and not sell, derivatives.

Exercise 5.33

Consider a portfolio that is long 1 share of the American put option in Section 4.2. Assume $p = 1/2$ and compute the expected value and the variance of the rate of return of the portfolio.

ANSWER: $\mathbb{E}[R] = 1/8$.

Exercise 5.34 (DO IT YOURSELF)

Repeat the previous exercise for the compound option in Exercise 3.3.

ANSWER: $\mathbb{E}[R] = 11/16$.

Solution to Exercise 5.22

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$$Y_{call} = \left[\left(\frac{1}{N+1}\sum_{t=0}^{N} S(t)\right) - K\right]_+$$

$$Y_{put} = \left[K - \left(\frac{1}{N+1}\sum_{t=0}^{N} S(t)\right)\right]_+$$

$$AC(0) = e^{-rN}\,\mathbb{E}_q[Y_{call}]$$

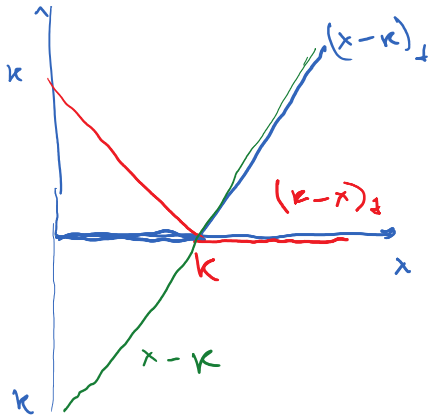
$$AP(0) = e^{-rN}\,\mathbb{E}_q[Y_{put}]$$

RISK-NEUTRAL PRICING FORMULA

$$AC(0) - AP(0) = \underbrace{e^{-rN}}\,\mathbb{E}_q[Y_{call}] - \underbrace{e^{-rN}}\,\mathbb{E}_q[Y_{put}]$$

$$= e^{-rN}\,\mathbb{E}_q[Y_{call} - Y_{put}] \leftarrow$$

REMEMBER THAT $(x-K)_+ - (K-x)_+ = x - K$ FOR ALL $x \in \mathbb{R}$



$$Y_{call} - Y_{put}$$

$$= \Big(\underbrace{\frac{1}{N+1}\sum_{t=0}^{N} S(t)}_{x} - K\Big)_+ - \Big(K - \frac{1}{N+1}\sum_{t=0}^{N} S(t)\Big)_+$$

$$= \frac{1}{N+1}\sum_{t=0}^{N} S(t) - K$$

$$AC(0) - AP(0) = e^{-rN}\,\mathbb{E}_q\left[\left(\frac{1}{N+1}\sum_{t=0}^{N} S(t)\right) - K\right]$$

$$= \underline{e^{-rN}}\left\{\left[\frac{1}{N+1}\sum_{t=0}^{N} S(t)\right] - \mathbb{E}_q[K]\right\}$$

$$= \underline{e} \left\{ \mathbb{E}_q \left[\frac{1}{N+1} \sum_{t=0}' S(t) \right] - \mathbb{E}_q [K] \right\}$$

$$= e^{-rN} \left(\frac{1}{N+1} \sum_{t=0}^{N} \mathbb{E}_q [S(t)] \right) - K e^{-rN}$$

NOW WE USE THAT $\mathbb{E}_q [S(t)] = \mathbb{E}_q [e^{rt} e^{-rt} S(t)] = e^{rt} \mathbb{E}_q [S^*(t)]$

$$= e^{rt} \mathbb{E}_q [S^*(0)] = S(0) e^{rt}$$

$$AC(0) - AP(0) = e^{-rN} \frac{1}{N+1} S(0) \sum_{t=0}^{N} e^{rt} - K e^{-rN}$$

WRITE $e^{rt} = (e^r)^t$ AND SET $\alpha = e^r$

IN THE FORMULA IN THE HINT

$$AC(0) - AP(0) = e^{-rN} \frac{S(0)}{N+1} \frac{1 - e^{r(N+1)}}{1 - e^r} - K e^{-rN}$$

$$= e^{-rN} \left(\frac{S(0)}{N+1} \frac{e^{r(N+1)} - 1}{e^r - 1} - K \right)$$

Solution of Exercise 5.32

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$$e^u = \frac{5}{4} \qquad e^d = \frac{1}{2} \qquad r = 0 \qquad S(0) = \frac{64}{25} \qquad p \in (0,1)$$

$$Y(t) = \left(\frac{11}{5} - S(t)\right)_+ \qquad \text{(AMERICAN PUT)}$$

$$\begin{tikzcd}[column sep=small]
 & & S(2) = 4 \\
 & S(1) = \frac{16}{5} \arrow[ur, "u"] \arrow[dr] & \\
S(0) = \frac{64}{25} \arrow[ur, "u"] \arrow[dr, "d"'] & & S(2) = 8/5 \\
 & S(1) = \frac{32}{25} \arrow[ur] \arrow[dr] & \\
 & & S(2) = \frac{16}{25}
\end{tikzcd}$$

$$\frac{11}{5} = \frac{55}{25}$$

$$\begin{tikzcd}[column sep=small]
 & & Y(2) = 0 = \hat{\pi}_Y(2) \\
 & \hat{\pi}_Y(1) = 1/5,\ Y(1) = 0 \arrow[ur, "u"] \arrow[dr, "d"'] & \\
Y(0) = 0,\ \hat{\pi}_Y(0) = \frac{11}{25} \arrow[ur, "u"] \arrow[dr, "d"'] & & Y(2) = 3/5 = \hat{\pi}_Y(2) \\
 & \boxed{Y(1) = \frac{23}{25},\ \hat{\pi}_Y(1) = \frac{23}{25}} \arrow[ur, "u"] \arrow[dr] & \\
 & & Y(2) = \frac{39}{25} = \hat{\pi}_Y(2)
\end{tikzcd}$$

$$\hat{\pi}_Y(t) = \max\left(Y(t),\ \overset{1}{e^{-r}}\left(q_u \hat{\pi}^u_Y(t+1) + q_d \hat{\pi}^d_Y(t+1)\right)\right) \qquad t = 0, 1$$

$$q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{1 - 1/2}{5/4 - 1/2} = \frac{1/2}{3/4} = 2/3 \qquad q_d = \frac{1}{3}$$

$$\hat{\pi}_Y(t) = \max\left(Y(t),\ \frac{2}{3}\hat{\pi}^u_Y(t+1) + \frac{1}{2}\hat{\pi}^d_Y(t+1)\right)$$

$$\hat{\Pi}_y(t) = \max\left(y(t),\ \frac{2}{3}\hat{\Pi}^u_y(t+1) + \frac{1}{3}\hat{\Pi}^d_y(t+1)\right)$$

IF $x_1 = u$

$$\hat{\Pi}_y(1) = \max\left(0,\ \frac{2}{3}\cdot 0 + \frac{1}{3}\cdot\frac{3}{5}\right) = 1/5$$

IF $x_1 = d$

$$\hat{\Pi}_y(1) = \max\left(\frac{23}{25},\ \frac{2}{3}\cdot\frac{3}{5} + \frac{1}{3}\,\frac{39}{25}\right)$$

$$= \max\left(\frac{23}{25},\ \frac{2}{5} + \frac{13}{25}\right) = \max\left(\frac{23}{25}, \frac{23}{25}\right) = \frac{23}{25}$$

$$\hat{\Pi}_y(0) = \max\left(0,\ \underbrace{\frac{2}{3}\,\frac{1}{5} + \frac{1}{3}\,\frac{23}{25}}\right) = \frac{11}{25}$$

IT IS ONLY OPTIMAL TO EXERCISE PRIOR TO MATURITY AT $t = 1$ IF THE STOCK PRICE GOES DOWN IN THE FIRST STEP

PART II

THE COMPOUND OPTION IN THIS EXERCISE IS A CALL ON THE AMERICAN PUT WITH MATURITY $T = 1$ AND STRIKE $K_1 = 8/25$. SO THE PAY-OFF FOR THIS COMPOUND OPTION IS

$$Z = \left(\hat{\Pi}_y(1) - 8/25\right)_+$$

$\Pi_Z(t)$, $t = 0, 1$ IS THE BINOMIAL PRICE OF THE COMPOUND OPTION

$$\Pi_Z(0) \xrightarrow{u} \Pi_Z(1) = \left(\frac{1}{5} - \frac{8}{25}\right)_+ = 0$$

$$\Pi_Z(0) \xrightarrow{d} \Pi_Z(1) = \left(\frac{23}{25} - \frac{8}{25}\right)_+$$

$$\Pi_Z(0) = 1/5$$

PRICE OF THE COMPOUND OPTION

$$= \left(\frac{23}{25} - \frac{8}{25}\right)_+$$
$$= 3/5$$

$$\Pi_Z(0) = e^{-r}\left(q_u \Pi_Z^u(1) + q_d \Pi_Z^d(1)\right)$$
$$= 1\left(\frac{2}{3}\cdot 0 + \frac{1}{3}\frac{3}{5}\right) = 1/5$$

$$h_S(t) = \frac{1}{S(t-1)} \frac{\Pi_Z^u(t) - \Pi_Z^d(t)}{e^u - e^d}$$

$$h_S(1) = h_S(0) = \frac{1}{S(0)} \frac{\Pi_Z^u(1) - \Pi_Z^d(1)}{e^u - e^d} = \frac{1}{\frac{64}{25}} \frac{-3/5}{5/4 - \frac{1}{2}}$$
$$= -\frac{5}{16}$$

$h_B(1)$? WE CAN COMPUTE IT USING THAT THE HEDGING PORTFOLIO REPLICATES THE DERIVATIVE, SO

$$V(0) = \Pi_Z(0)$$
$$\hookrightarrow h_S(0)S(0) + h_B(0)B(0) = \Pi_Z(0)$$
$$-\frac{5}{16}\cdot\frac{64}{25} + h_B(0) = \frac{1}{5}$$
$$\Rightarrow h_B(0) = \frac{1}{5} + \frac{4}{5} = 1 = h_B(1)$$

PART III

LET'S COMPUTE THE RETURN ON THE COMPOUND OPTION ALONG ALL POSSIBLE PATHS:

$R(u,d) = R(u,u) = -1/5$ WITH PROBABILITY $p(1-p) + p^2 = p$

$R(d,d) = R(d,u) = \frac{23}{25} - \frac{1}{5}\left(-\frac{8}{25}\right)$

$= \frac{23}{25} - \frac{13}{25} = \frac{10}{25} = \frac{2}{5}$ (IF THE AMERICAN PUT IS EXERCISED)

WITH PROBABILITY $1-p$

SO, THE EXPECTED RETURN IF THE AMERICAN PUT IS EXERCISED IMMEDIATELY IS

$$E_1[R] = -\frac{1}{5}p + \frac{2}{5}(1-p) = \boxed{-\frac{3}{5}p + \frac{2}{5}}$$

IF THE AMERICAN PUT IS NOT EXERCISED AT TIME $t=1$, THEN

$R(d,d) = \frac{39}{25} - \frac{1}{5} - \frac{8}{25} = \frac{39}{25} - \frac{13}{25} = \frac{26}{25}$ PROB. $(1-p)^2$

$R(d,u) = \frac{3}{5} - \frac{1}{5} - \frac{8}{25} = \frac{3}{5} - \frac{13}{25} = \frac{2}{25}$ PROB $p(1-p)$

SO IN THIS CASE, THE EXPECTED RETURN IS

$$E_2[R] = \underbrace{\frac{26}{25}(1-p)^2} + \underbrace{\frac{2}{25}(1-p)p} - \frac{1}{5}p = \boxed{\frac{1}{25}(3p-2)(8p-13)}$$

QUESTION: WHEN IS $E_1[R] > E_2[R]$?

$\underline{E_1[R]} > E_2[R]$ IF $\frac{2}{3} < p < 1$

$E_1[R] < \underline{E_2[R]}$ IF $0 < p < \frac{2}{3}$

$E_1[R] = E_2[R]$ IF $p = 2/3$

Solution to Exercise 5.33

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AMERICAN PUT OPTION IN SECTION 4.2

$$N = 2, \quad e^u = \frac{7}{4} \quad e^d = \frac{1}{2} \quad e^r = \frac{9}{8} \quad p = 1/2$$

$$Y(t) = \left(\frac{3}{4} - S(t)\right)_+ \qquad \text{STRIKE } 3/4$$

$$\begin{tikzcd}
 & & \hat{\Pi}_Y(2) = 0 \\
 & \hat{\Pi}_Y(1) = 0 \arrow[ur, "u"] \arrow[dr, "d"'] & \\
\hat{\Pi}_Y(0) = 1/9 \arrow[ur, "u"] \arrow[dr, "d"'] & & \hat{\Pi}_Y(2) = 0 \\
 & \boxed{\hat{\Pi}_Y(1) = \frac{1}{4}} \arrow[ur, "u"] \arrow[dr, "d"'] & \\
 & & \hat{\Pi}_Y(2) = 1/2
\end{tikzcd}$$

OPTIMAL EXERCISE TIME

EXPECTED VALUE OF THE RATE OF RETURN

$$\underbrace{R(u,u)}_{1/4} = \underbrace{R(u,d)}_{1/4} = -1/9 \quad \text{WITH PROB } 1/2$$

$$R(d,u) = R(d,d) = \frac{1}{4} - \frac{1}{9} = \frac{5}{36} \quad \text{WITH PROB } 1/2$$

IF THE AMERICAN PUT IS EXERCISE AT TIME $t=1$

SO THE EXPECTED RETURN IF THE AMERICAN PUT IS EXERCISED OPTIMALLY WOULD BE

$$E[R] = \frac{1}{2}\cdot\left(-\frac{1}{9}\right) + \frac{5}{36}\cdot\frac{1}{2}$$

AND THE EXPECTED RATE OF RETURN IS

$$E[R/\hat{\Pi}_Y(0)] = \frac{1}{\hat{\Pi}_Y(0)} E[R] = 9\left(\frac{1}{2}\cdot\left(-\frac{1}{9}\right) + \frac{5}{36}\cdot\frac{1}{2}\right)$$

$$E\left[R/\hat{\pi}_y(0)\right] = \frac{1}{1/9} E[R] = 9\left(\frac{1}{2}\cdot\left(-\frac{1}{9}\right) + \frac{5}{36}\cdot\frac{1}{2}\right)$$

$$= \left(-\frac{1}{2} + \frac{5}{8}\right) = \frac{1}{8}$$

IF THE AMERICAN PUT IS NOT EXERCISED, THEN

$R(d,u) = -\frac{1}{9}$ WITH PROB $1/4$

$R(d,d) = \frac{1}{2} - \frac{1}{9} = \frac{7}{18}$ WITH PROB. $1/4$

SO IN THIS CASE

$$E[R] = -\frac{1}{9}\cdot\frac{1}{4} + \frac{7}{18}\cdot\frac{1}{4} - \frac{1}{2}\cdot\frac{1}{9}$$

THE EXPECTED RATE OF RETURN IS

$$E[R]/\pi_y(0) = 9\left(-\frac{1}{9}\cdot\frac{1}{4} + \frac{7}{18}\cdot\frac{1}{4} - \frac{1}{2}\cdot\frac{1}{9}\right)$$

$$= -\frac{1}{4} + \frac{7}{8} - \frac{1}{2} = \frac{-2+7-4}{8} = \frac{1}{8}$$

$$VAR[R] = R(u,u)^2 p^2 + R(u,d)^2 p(1-p) + R(d,u)^2 p(1-p) + R(d,d)^2 (1-p)^2 - E[R]^2$$

$\parallel$

$E[R^2] - E[R]^2$

Solution to Exercise 5.23

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WANT TO PROVE THAT $\{\Pi^*_Y(t)\}_{t=0,\dots,N}$ IS

MARTINGALE IN THE RISK-NEUTRAL PROBABILITY:

$$\mathbb{E}_q\left[\Pi^*_Y(t) \mid \Pi^*_Y(0),\dots,\Pi^*_Y(t-1)\right] = \Pi^*_Y(t-1) \quad (1)$$

FIRST WE SHOW THAT (1) FOLLOWS FROM

$$\mathbb{E}_q\left[\Pi^*_Y(t) \mid S(0),\dots S(t-1)\right] = \Pi^*_Y(t-1) \quad (2)$$

PROOF: SUPPOSE (2) HOLDS AND TAKE THE CONDITIONAL EXPECTATION W.R.T. $\Pi^*_Y(0),\dots,\Pi^*_Y(t-1)$ OF BOTH SIDES OF (2):

LEFT HAND SIDE:

$$\mathbb{E}_q\left[\mathbb{E}_q\left(\Pi^*_Y(t) \mid S(0),\dots S(t-1)\right) \mid \Pi^*_Y(0),\dots \Pi^*_Y(t-1)\right]$$

$$= \{\text{TOWER PROPERTY}\} = \mathbb{E}_q\left[\Pi^*_Y(t) \mid \Pi^*_Y(0),\dots,\Pi^*_Y(t-1)\right]$$

RIGHT HAND SIDE

$$\mathbb{E}_q\left[\Pi^*_Y(t-1) \mid \Pi^*_Y(0),\dots \Pi^*_Y(t-1)\right] = \Pi^*_Y(t-1)$$

HENCE (2) $\Rightarrow$ (1)

SO TO PROVE (1) IT IS SUFFICIENT TO PROVE (2)

WE NEED THE RISK-NEUTRAL PRICING FORMULA AT TIME t:

$$\Pi_Y(t) = e^{-r(N-t)}\,\mathbb{E}_q\left[Y \mid S(0),\dots S(t)\right] \quad (*)$$

SUBSTITUTING THIS IN THE L.H.S. OF (2) WE GET

$$\mathbb{E}_q\left[e^{-rt}\Pi_Y(t) \mid S(0),\dots S(t-1)\right]$$

$$= e^{-rt}\,\mathbb{E}_q\left[e^{-r(N-t)}\mathbb{E}_q[Y \mid S(0),\dots S(t)] \mid S(0),\dots S(t-1)\right]$$

THIS CONTAINS LESS INFORMATION

$$= \{\text{TOWER PROPERTY}\} = e^{-rN}\,\mathbb{E}_q[Y \mid S(0),\dots S(t-1)]$$

$$= \underbrace{e^{-rN}\;e^{r(N-(t-1))}}\left(e^{-r(N-(t-1))}\,\mathbb{E}_q[Y \mid S(0),\dots S(t-1)]\right)$$

{REPLACING t WITH $t-1$ IN (*) WE OBTAIN THAT THIS IS $\Pi_Y(t-1)$}

$$= e^{-r(t-1)}\,\Pi_Y(t-1) = \underline{\Pi^*_Y(t-1)} \quad \boxtimes$$

FOR THE SECOND QUESTION (RECURRENCE FORMULA) WE WANT TO PROVE THAT

$$\Pi_Y(t) = e^{-r}\left(q_u\,\Pi^u_Y(t+1) + q_d\,\Pi^d_Y(t+1)\right)$$

USING THE RISK-NEUTRAL PRICING FORMULA AND THE DEFINITION OF THE CONDITIONAL EXPECTATION:

$$\Pi_Y(t) = e^{-r(N-t)}\,\mathbb{E}_q[Y \mid S(0),\dots S(t)] = \left\{ S(t) = \begin{cases} S(t-1)e^u \\ S(t-1)e^d \end{cases} \right\}$$

$$= e^{-r(N-t)}\Big(\mathbb{E}_q[Y \mid S(0),\dots S(t), S(t+1)=S(t)e^u]\,\underbrace{\mathbb{P}_q(S(t+1)=S(t)e^u)}_{q_u}$$

$$+\ \mathbb{E}_q[Y \mid S(0),\dots S(t), S(t+1)=S(t)e^d]\,\underbrace{\mathbb{P}_q(S(t+1)=S(t)e^d)}_{q_d}\Big)$$

$$= e^{-r(N-t)}\Big[e^{r(N-(t+1))}\, e^{-r(N-(t+1))}\, E_q\left[Y \mid S(0),\ldots,S(t),\, S(t+1)=S(t)e^{u}\right] q_u$$

$$+\; e^{r(N-(t+1))}\, e^{-r(N-(t+1))}\, E_q\left[Y \mid S(0),\ldots,S(t),\, S(t+1)=S(t)e^{d}\right] q_d \Big]$$

$$= e^{-r}\left[q_u\, \pi_Y^u(t+1) + \pi_Y^d(t+1)\, q_d \right]$$

THIRD QUESTION:

SHOW THAT THE MARKET $(S(t), B(t), \pi_Y(t))$ IS FREE OF SELF-FINANCING ARBITRAGES.

SINCE $S^*(t), \pi_Y^*(t)$ ARE MARTINGALE IN THE RISK-NEUTRAL PROBABILITY THE DISCOUNTED VALUE $V^*(t)$ OF ANY SELF-FINANCING PORTFOLIO INVESTED IN THIS MARKET IS ALSO A MARTINGALE IN THE RISK-NEUTRAL PROBABILITY. IN PARTICULAR, SINCE MARTINGALES HAVE CONSTANT EXPECTATION, THEN

$$E_q[V^*(N)] = E_q[V^*(0)] = V(0)$$

ASSUME THE PORTFOLIO IN AN ARBITRAGE. THEN $V(0) = 0$. HENCE $E_q[V^*(N)] = e^{-rN} E_q[V(N)] = 0$

MOREOVER $V(N) \geq 0$, AND SO ALSO $V^*(N) \geq 0$

SO WE HAVE $E_q[V(N)] = 0$ AND $V(N) \geq 0$

AND THEREFORE $V(N) = 0$, AND SO THE PORTFOLIO IS NOT AN ARBITRAGE.