Lecture_21

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Options and Mathematics: Lecture 21

December 8, 2020

Black-Scholes price of European call and put options (1973, BLACK-SCHOLES MERTON)

In this lecture we focus the discussion on call/put options. We then assume that the pay-off of the derivative is given by

$$Y = (S(T) - K)_+, \text{ i.e., } Y = g(S(T)),\ g(z) = (z - K)_+, \quad \text{for a call option,}$$

$$Y = (K - S(T))_+, \text{ i.e., } Y = g(S(T)),\ g(z) = (K - z)_+, \quad \text{for a put option.}$$

As usual, we denote by $C(t, S(t), K, T)$ the value at time t of the call option with strike K and maturity T and by $P(t, S(t), K, T)$ the value of the corresponding put option. $v(t,x) = C(t,x,K,T)$ FOR CALLS AND $v(t,x) = P(t,x,K,T)$ FOR PUT OPTION

Recall that the Black-Scholes price at time t of the standard European derivative with payoff function g and maturity T is

$$\Pi_Y(t) = v(t, S(t)), \quad v(t,x) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\mathbb{R}} g\left(x e^{(r - \frac{\sigma^2}{2})\tau} e^{\sigma\sqrt{\tau}\, y}\right) e^{-\frac{y^2}{2}}\, dy \quad \tau = T - t.$$

IMP! YOU HAVE TO KNOW IT FOR THE EXAM

Theorem 6.14

The Black-Scholes price at time t of the European call option with strike price $K > 0$ and maturity $T > 0$ is given by $C(t, S(t), K, T)$, where

$$\boxed{C(t, x, K, T) = x\Phi(d_{(+)}) - Ke^{-r\tau}\Phi(d_{(-)})} \quad (1)$$

$$\boxed{d_{(\pm)} = \frac{\log\left(\frac{x}{K}\right) + \left(r \pm \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}}$$

$d_{(+)} = d_{(-)} + \sigma\sqrt{\tau}$

and where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\, dy$$

THIS FUNCTION APPEARS IN MANY MANY APPLICATIONS OF PROB. THEORY.

is the standard normal distribution.

The Black-Scholes price of the corresponding put option is given by $P(t, S(t), K, T)$, where

$$\boxed{P(t, x, K, T) = \Phi(-d_{(-)})Ke^{-r\tau} - \Phi(-d_{(+)})x} \quad (2)$$

Moreover the put-call parity identity holds:

$$C(t, S(t), K, T) - P(t, S(t), K, T) = S(t) - Ke^{-r\tau}.$$

USING (1) AND (2)

$$v(t,x) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\mathbb{R}} g\Big(\underbrace{x\, e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}\, y}}_{z}\Big)\, e^{-\frac12 y^2}\, dy$$

$\equiv$ $C(t,x,K,T)$

WHEN $g(z) = (z-K)_+$

Proof. We derive the Black-Scholes price of call options only, the argument for put options being similar. We substitute $g(z) = (z-K)_+$ into $v(t,x)$ and obtain

$$v(t,x) = C(t,x,K,T) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(x e^{(r-\frac12\sigma^2)\tau} e^{\sigma\sqrt{\tau}y} - K \right)_+ e^{-\frac{y^2}{2}}\, dy.$$

Now we use that $\log x + (r-\frac12\sigma^2)\tau + \sigma\sqrt{\tau}\, y > \log K \iff y > \dfrac{\log\frac{K}{x} - (r-\frac12\sigma^2)\tau}{\sigma\sqrt{\tau}} = -\dfrac{\log\frac{x}{K} + (r-\frac12\sigma^2)\tau}{\sigma\sqrt{\tau}}$

$$\dfrac{\log\frac{x}{K} + (r-\frac12\sigma^2)\tau}{\sigma\sqrt{\tau}} = d_{(-)}$$

$$x e^{(r-\frac12\sigma^2)\tau} e^{\sigma\sqrt{\tau}y} > K \quad \text{if and only if } y > -d_{(-)}.$$

Hence $\dfrac{e^{-r\tau}}{\sqrt{2\pi}} \displaystyle\int_{y>-d(-)} \left(x e^{(r-\frac12\sigma^2)\tau + \sigma\sqrt{\tau}y} - K \right) e^{-\frac12 y^2}\, dy$

$$C(t,x,K,T) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \left[\int_{-d_{(-)}}^{\infty} x e^{\cancel{(r-\frac12\sigma^2)\tau}} e^{\sigma\sqrt{\tau}y} e^{-\frac{y^2}{2}} - K \int_{-d_{(-)}}^{\infty} e^{-\frac{y^2}{2}}\, dy \right].$$

Using $e^{\sigma\sqrt{\tau}y - \frac12 y^2}$

$$-\frac12\left(y^2 - 2\sigma\sqrt{\tau}\,y\right) = -\frac12\left(y^2 - 2\sigma\sqrt{\tau}\,y + \sigma^2\tau\right) + \frac{\sigma^2\tau}{2}$$

$$(\ldots) = (y - \sigma\sqrt{\tau})^2$$

$$-\frac12 y^2 + \sigma\sqrt{\tau} y = -\frac12 (y - \sigma\sqrt{\tau})^2 + \frac{\sigma^2}{2}\tau$$

and changing variable in the integrals we obtain

$z = y - \sigma\sqrt{\tau}$ $\quad dy = dz$

$y = -z$

$$C(t,x,K,T) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \left[x e^{r\tau} \int_{-d_{(-)}}^{\infty} e^{-\frac12 (y-\sigma\sqrt{\tau})^2}\, dy - K \int_{-d_{(-)}}^{\infty} e^{-\frac{y^2}{2}}\, dy \right] =$$

$$= \frac{e^{-r\tau}}{\sqrt{2\pi}} \left[\int_{-d_{(-)}-\sigma\sqrt{\tau}}^{\infty} e^{-\frac12 z^2}\, dz - K \int_{-d_{(-)}}^{\infty} e^{-\frac{y^2}{2}}\, dy \right]$$

$$= \frac{e^{-r\tau}}{\sqrt{2\pi}} \left[x e^{r\tau} \int_{-\infty}^{d_{(-)}+\sigma\sqrt{\tau}} e^{-\frac12 y^2}\, dy - K \int_{-\infty}^{d_{(-)}} e^{-\frac{y^2}{2}}\, dy \right]$$

$$= x\Phi(d_{(+)}) - K e^{-r\tau}\Phi(d_{(-)}),$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\, dy$$

where we used that $d_{(+)} = d_{(-)} + \sigma\sqrt{\tau}$.

$C(t,x,K,T) - P(t,x,K,T) = x - Ke^{-r\tau}$ FOR ALL $x>0$

REPLACE $x = S(t)$ AND GET THE PUT-CALL PARITY

As to the put-call parity, we have

$$\begin{aligned} C(t,x,K,T) - P(t,x,K,T) &= \overbrace{x\Phi(d_{(+)}) - Ke^{-r\tau}\Phi(d_{(-)})}^{C} - \overbrace{(\Phi(-d_{(-)})Ke^{-r\tau} - x\Phi(-d_{(+)}))}^{P} \\ &= x\underbrace{(\Phi(d_{(+)}) + \Phi(-d_{(+)}))}_{1} - Ke^{-r\tau}\underbrace{(\Phi(d_{(-)}) + \Phi(-d_{(-)}))}_{1}. \end{aligned}$$

As $\Phi(z) + \Phi(-z) = 1$, the claim follows. FOR ALL $z \in \mathbb{R}$ □

Next we construct replicating, and thus hedging, portfolio processes for call and put options.

Theorem 6.15 NOT IN THE LIST FOR THE EXAM

The following are self-financing replicating portfolio processes for European call/put options on Black-Scholes markets:

$\Phi(x) = \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}}$

REPLACE $x = d_{(+)}$

$\Phi(d_{(+)}) = \int_{-\infty}^{d_{(+)}} e^{-\frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}}$

THE FORMULA FOR $h_B(t)$ FOLLOWS BY THE REPLICATING CONDITION: $h_S(t)S(t) + h_B(t)B(t) = \Pi_Y(t)$

$$h_S(t) = \Phi(d_{(+)}) > 0, \quad h_B(t) = -\frac{Ke^{-r\tau}\Phi(d_{(-)})}{B(t)} \quad \text{for call options}$$

$$h_S(t) = -\Phi(-d_{(+)}) < 0, \quad h_B(t) = \frac{Ke^{-r\tau}\Phi(-d_{(-)})}{B(t)} \quad \text{for put options.}$$

$h_S(t) = \Delta(t, S(t))$, $\Delta(t,x) = \partial_x \Pi(t,x) = \begin{cases} \partial_x C(t,x,K,T) & \text{CALLS} \\ \partial_x P(t,x,K,T) & \text{PUTS} \end{cases}$

The greeks

(ON THURSDAY)

The Black-Scholes price of call and put options depends on the price of the underlying stock, the time to maturity, the strike price, as well as on the (constant) market parameters r, σ (it does not depend on α).

The partial derivatives of the pricing function with respect to these variables are called **greeks**. We collect the most important ones (for call options) in the following theorem.

$C(t, S(t), K, T) \equiv$ PRICE AT TIME t OF THE CALL WITH STRIKE K AND MATURITY T.

IN BLACK-SCHOLES THEORY: $S(t) = S_0 e^{\alpha t + \sigma W(t)}$ AND

$C(t, x, K, T) = x\,\Phi(d_{(+)}) - Ke^{-r\tau}\Phi(d_{(-)})$ $\quad \tau = T - t$

$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-1/2 y^2} dy$, $\quad d_{(\pm)} = \left(\log\frac{x}{K} + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right) / \sigma\sqrt{\tau}$

$C(t, x, K, T)$ DEPENDS ALSO ON r AND ON σ

$h_S(t) = \Delta(t, S(t))$

$\Delta(t, x) = \partial_x C$

Theorem 6.16 (the greeks)

The pricing function $C(t, x, K, T)$ of call options satisfies the following:

DELTA $\to$ $\Delta := \partial_x C = \Phi(d_{(+)})$, > 0

GAMMA $\to$ $\Gamma := \partial_x^2 C = \dfrac{\phi(d_{(+)})}{x\sigma\sqrt{\tau}}$, > 0

RHO $\to$ $\rho := \partial_r C = K\tau e^{-r\tau}\Phi(d_{(-)})$, > 0

THETA $\to$ $\Theta := \partial_t C = -\dfrac{x\phi(d_{(+)})\sigma}{2\sqrt{\tau}} - rKe^{-r\tau}\Phi(d_{(-)})$, < 0 (HENCE $\partial_T C > 0$)

VEGA $\to$ $\nu := \partial_\sigma C = x\phi(d_{(+)})\sqrt{\tau}$ (called "vega"), > 0

where we recall that $\phi(z) = \Phi'(z) = (\sqrt{2\pi})^{-1}e^{-\frac{z^2}{2}}$. In particular:

- $\Delta > 0$, i.e., the price of a call is increasing on the price of the underlying stock;
- $\Gamma > 0$, i.e., the price of a call is convex on the price of the underlying stock;
- $\rho > 0$, i.e., the price of the call is increasing on the interest rate of the risk-free asset;
- $\Theta < 0$, i.e., the price of the call is decreasing in time;
- $\nu > 0$, i.e., the price of the call is increasing on the volatility of the stock.

The greeks measure the sensitivity of option prices with respect to the market conditions. This information can be used to draw some important conclusions.

For instance, since vega is positive, then the wish of an investor with a long position on a call option is that the volatility of the underlying stock increased. As usual, since this might not happen, the buyer of the call may incur in a loss if the stock volatility decreases (since the call option will loose value). This exposure to volatility can be secured by adding volatility swaps into the portfolio.

Implied volatility

Let $C(t, S(t), K, T)$ denote the Black-Scholes price of the European call with strike price K, maturity time T on a stock with price $S(t)$ at time t.

Recall that in the derivation of the Black-Scholes price it is assumed that the price of the stock follows the geometric Brownian motion

$$S(t) = S(0)e^{\alpha t + \sigma W(t)},$$

where $\{W(t)\}_{t\in[0,T]}$ is a Brownian motion stochastic process, $\sigma > 0$ is the instantaneous volatility and $\alpha \in \mathbb{R}$ is the instantaneous mean of log-return. The function $C(t, x, K, T)$ is given by

$$C(t, x, K, T) = x\Phi(d_{(+)}) - Ke^{-r\tau}\Phi(d_{(-)}),$$

where $r > 0$ is the (constant) risk-free rate of the money market, $\tau = T - t$ is the time left to the expiration of the call,

$$d_{(\pm)} = \frac{\log\frac{x}{K} + (r \pm \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}},$$

and Φ denotes the standard normal distribution,

$$\Phi(z) = \int_{-\infty}^{z} e^{-\frac{y^2}{2}}\,\frac{dy}{\sqrt{2\pi}}.$$

Remarkably, $C(t, S(t), K, T)$ does not depend on the mean of log-return α of the stock price.

However it depends on the parameters (σ, r) and since here we are particularly interested in the dependence on the volatility, we re-denote the Black-Scholes price of the call as

$$C(t, S(t), K, T, \sigma).$$

Moreover

$$\frac{\partial C}{\partial \sigma} = \text{vega} = x\phi(d_{(+)})\sqrt{\tau} > 0.$$

Hence the Black-Scholes price of the option is an increasing function of the volatility. Furthermore,

$$\lim_{\sigma \to 0^+} C(t, S(t), K, T, \sigma) = (S(t) - Ke^{-r\tau})_+, \quad \lim_{\sigma \to +\infty} C(t, S(t), K, T, \sigma) = S(t),$$

see Exercise 6.12.

Therefore the function $C(t, S(t), K, T, \cdot)$ is a one-to-one map from $(0, \infty)$ into the interval $((S(t) - Ke^{-r\tau})_+, S(t))$, see Figure 6.1. This property makes it possible to define the concept of implied volatility as follows.

Defintion 6.19 (implied volatility) YOU HAVE TO KNOW IT FOR THE EXAM

Let $\widetilde{C}(t)$ be the market price at time $t < T$ of the European call with strike K and maturity T. If $\widetilde{C}(t) \in ((S(t) - Ke^{-r\tau})_+, S(t))$, the **implied volatility** σ_{imp} of the call option is the unique value of the volatility parameter σ such that

$$\boxed{C(t, S(t), K, T, \sigma_{\text{imp}}) = \widetilde{C}(t)}$$

The implied volatility must be computed numerically (for instance with the function `blsimpv` in Matlab), since there is no close formula for it.

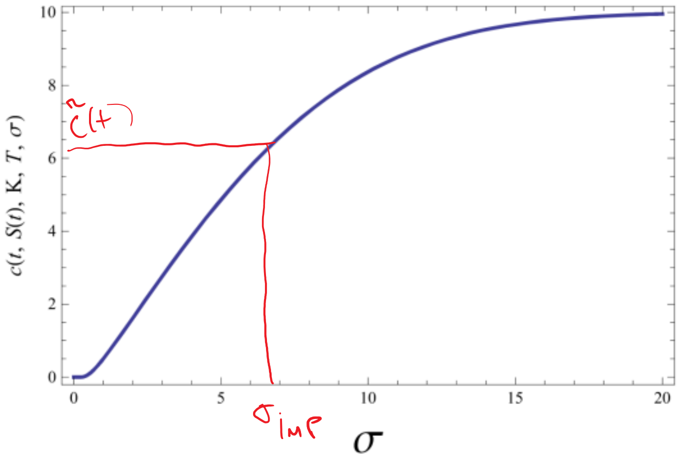


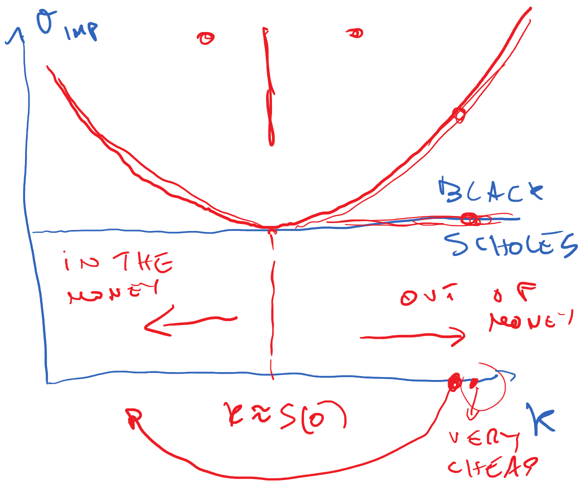
Figure 1: We fix $S(t) = 10$, $K = 12$, $r = 0.01$, $\tau = 1/12$ and depict the Black-Scholes price of the call as a function of the volatility.

The implied volatility of an option (in this example of a call option) is a very important parameter and it is usually quoted together with the price of the option.

If the market followed exactly the assumptions in the Black-Scholes theory, then the implied volatility would be constant (independent of time) and it would be the same for all call options on the same stock with the same strike and maturity.

However for real market options this turns out to be false, i.e., the implied volatility depends on time, K and T. In this respect, σ_{imp} may be viewed as a quantitative measure of how real markets deviate from ideal Black-Scholes markets.

VOLATILITY SMILE



Volatility smile

As mentioned before, the implied volatility in real markets depends on the parameters K, T. Here we are particularly interested in the dependence on the strike price, hence we re-denote the implied volatility as $\sigma_{\text{imp}}(K)$.

If the market behaved exactly as in the Black-Scholes theory, then $\sigma_{\text{imp}}(K) = \sigma$ for all values of K, hence the graph of the function $K \to \sigma_{\text{imp}}(K)$ would be a straight horizontal line.

Given that real markets do not satisfy exactly the assumptions in the Black-Scholes theory, what can we say about the graph of the function $K \to \sigma_{\text{imp}}(K)$?

Remarkably, it has been found that there exists recurrent convex shapes for the graph of this function, which are known as **volatility smile** and **volatility skew**, see Figures 2-3.

The minimum of the volatility smile is reached at the strike price $K \approx S(t)$, i.e., when the call is nearly at the money.

This behavior indicates that the more the call is far from being at the money, the more it will be overpriced. Volatility smiles and skews have been found in the market especially after the crash in 1987 (Black Monday), indicating that this event led investors to be more cautious when trading on options that are in or out of the money.

Devise mathematical models of stochastic volatility and asset prices able to reproduce volatility curves is an active research topic in mathematical finance.

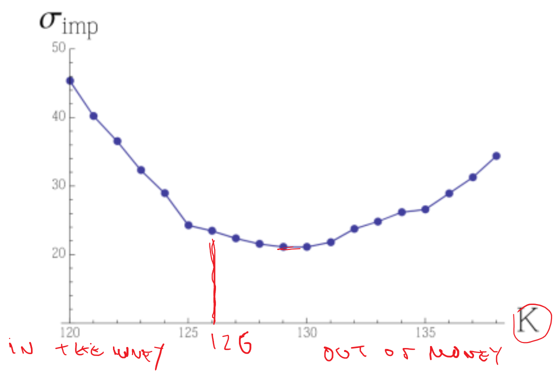


Figure 2: Volatility Smile of a call option on Apple expiring May 15^{th}, 2015. The data were taken on May 12^{th}, when the Apple stock quoted 126.34 dollars.

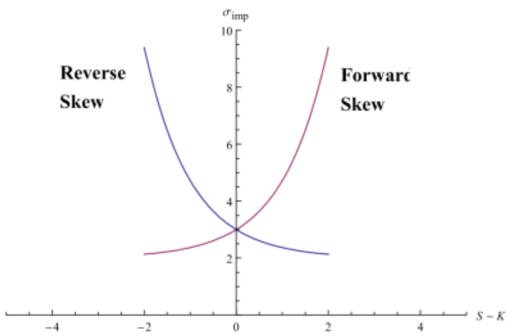


Figure 3: Volatility skews (not from real data!)

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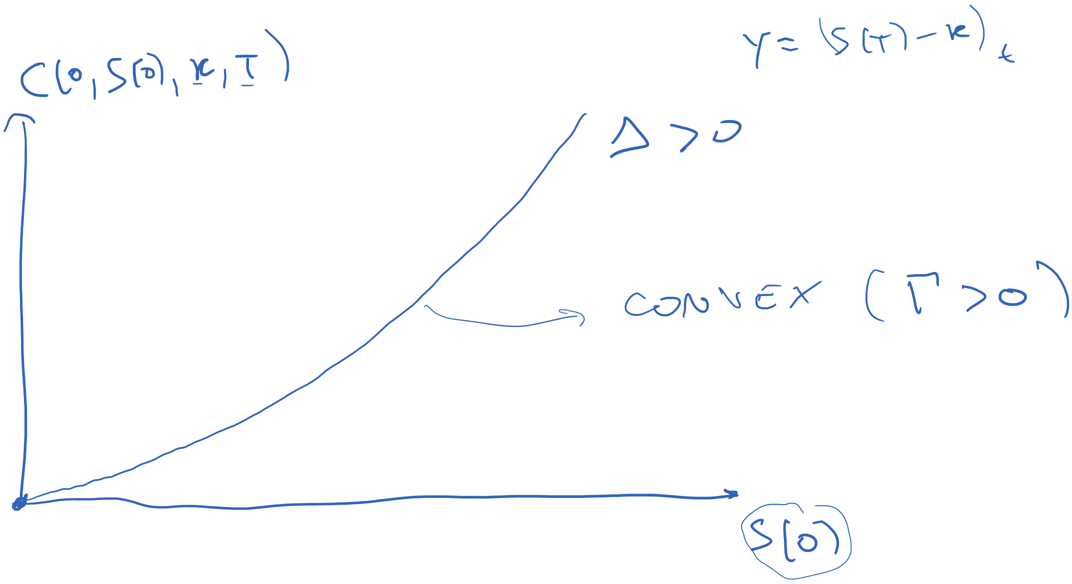

$C(0, S_0, K, T) \leq S_0$

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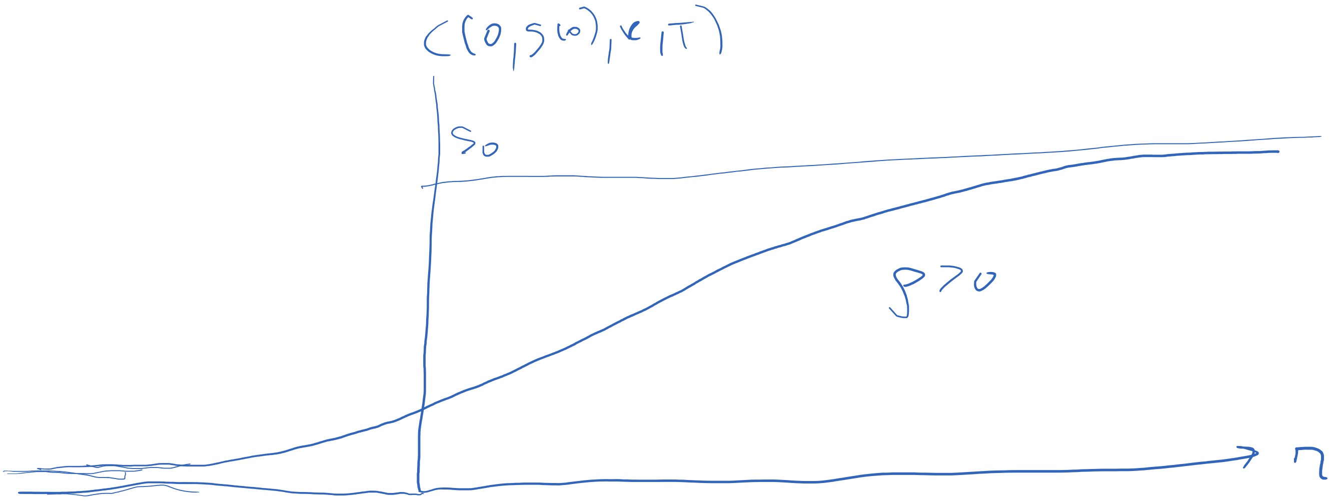

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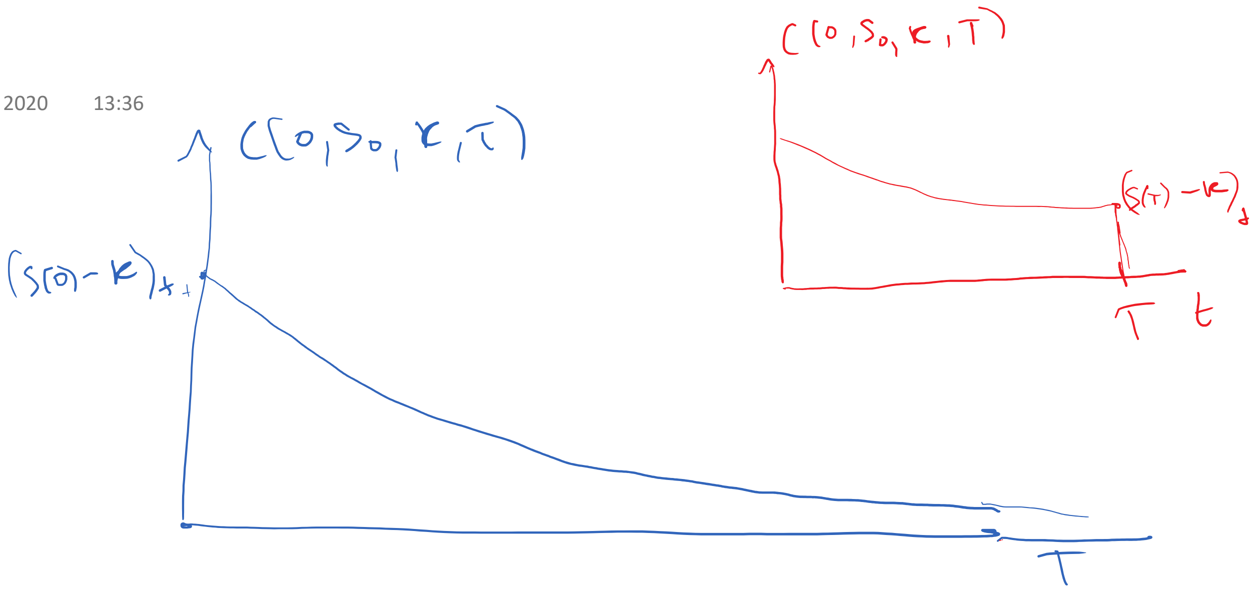

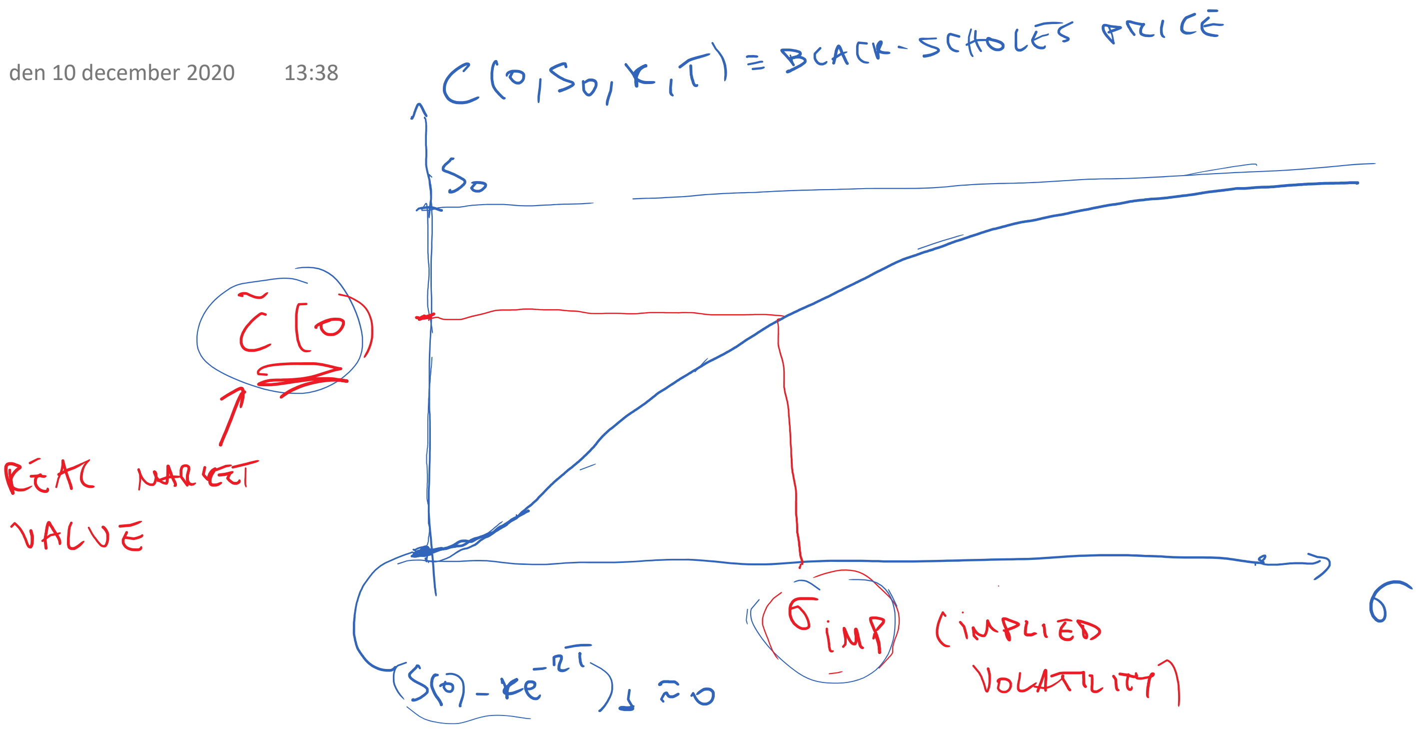
C(0, S0, K, T) = BLACK-SCHOLES PRICE
S0
C̃(0)
REAL MARKET VALUE
σ
σ IMP (IMPLIED VOLATILITY)
(S(0) - Ke^-rT)+ ≈ 0