Lecture_23

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Options and Mathematics: Lecture 23

December 10, 2020

The Asian option

For European call and put options, and for other simple standard European derivatives, the Black-Scholes pricing formula can be reduced to a simple expression in terms of the standard normal distribution.

For non-standard derivatives, i.e., when the pay-off depends on the price of the stock at different times (and not only at maturity), this reduction is in general not possible.

Nevertheless the risk-neutral pricing formula can be used to compute numerically the Black-Scholes price of non-standard derivatives using the so called **Monte Carlo method**.

We illustrate the procedure in the important case of the Asian option.

Recall that the Asian call, resp. put, option in the time-continuum case is defined as the non-standard European derivative with pay-off

CASH-SETTLED OTC OPTIONS

NON-STANDARD DERIVATIVES

$$Y_{\text{call}} = \left(\frac{1}{T}\int_0^T S(t)\,dt - K\right)_+, \quad \text{resp.} \quad Y_{\text{put}} = \left(K - \frac{1}{T}\int_0^T S(t)\,dt\right)_+,$$

ARITHMETIC AVERAGE OF S(t) IN THE INTERVAL [0, T]

where $K > 0$ is the strike price of the option.

THE BLACK-SCHOLES PRICE AT TIME $t=0$ IS

$$\Pi_{AC}(0) = e^{-rT}\,\mathbb{E}_q[Y_{\text{call}}] \qquad \Pi_{AP}(0) = e^{-rT}\,\mathbb{E}_q[Y_{\text{put}}]$$

FOR THE STANDARD CALL THE PAY-OFF IS

$(S(T)-K)_+$

The Black-Scholes price at time $t = 0$ of these options are given respectively by

$$\Pi_{\rm AC}(0) = e^{-rT}\mathbb{E}_q[Y_{\rm call}], \quad \Pi_{\rm AP}(0) = e^{-rT}\mathbb{E}_q[Y_{\rm call}].$$

Exercise 6.23

Derive the following put-call parity identity:

$$\Pi_{\rm AC}(0) - \Pi_{\rm AP}(0) = e^{-rT}\left(\frac{e^{rT}-1}{rT}S_0 - K\right).$$

$\frac{\sum_{i=1}^{N} x_i}{N}$ = ARITHMETIC AVERAGE OF $x_1,\dots,x_N$

$\left(\prod_{i=1}^{N} x_i\right)^{1/N}$ = GEOMETRIC AVERAGE OF $x_1 \dots x_N$

Exercise 6.24

ALSO EXIST IN REAL MARKETS (OTC)

The Asian call with geometric average is the non-standard European derivative with pay-off

$$Q = \left(e^{\frac{1}{T}\int_0^T \log S(t)\,dt} - K\right)_+.$$

GEOMETRIC AVERAGE

THIS IS MORE SIMILAR TO A STANDARD CALL

Show that the Black-Scholes price at time $t = 0$ of this derivative is given by

$$\Pi^{(G)}_{\rm AC}(0) = e^{-rT}(e^{qT}S_0\Phi(d_1) - K\Phi(d_2))$$

where

$$q = \frac{1}{2}(r - \frac{\sigma^2}{6})T, \quad d_2 = d_1 - \sigma\sqrt{\frac{T}{3}}, \quad d_1 = \frac{\log\frac{S_0}{K} + \frac{1}{2}(r + \frac{\sigma^2}{6})T}{\sigma\sqrt{T/3}}.$$

Derive also the analogous formula the Black-Scholes price of the put option as well as the corresponding put-call parity.

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HINT: You need Theorem 6.6 (and not 6.9 as written in the book!).

BLACK-SCHOLES PRICE AT $t=0$ OF THE EUROPEAN DERIVATIVE WITH PAY Y AND MATURITY $T>0$ IS GIVEN BY THE RISK-NEUTRAL PRICING FORMULA $\Pi_Y(0) = e^{-rT}\mathbb{E}_q[Y]$

IF $Y = g(S(T))$, THIS PRICE CAN BE WRITTEN IN THE INTEGRAL FORM $\Pi_Y(0) = v_0(S_0)$, $v_0(x) = e^{-rT}\int_{\mathbb{R}} g\left(x e^{(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}y}\right)\frac{e^{-\frac{1}{2}y^2}dy}{\sqrt{2\pi}}$

The Monte Carlo method

The Monte Carlo method is, in its simplest form, a numerical method to compute the expectation of a random variable.

Its mathematical validation is based on the **Law of Large Numbers**, which states the following: Suppose $\{X_i\}_{i\geq 1}$ is a sequence of i.i.d. random variables with expectation $\mathbb{E}[X_i] = \mu$. Then the sample average of the first n components of the sequence, i.e.,

I.I.D. INDEPENDENT IDENTICALLY DISTRIBUTED

$$\overline{X}_n = \frac{1}{n}(X_1 + X_2 + \cdots + X_n),$$

$\mathbb{P}(|\overline{X}_n - \mu| > \varepsilon) \to 0$ FOR ALL $\varepsilon > 0$

converges (in probability) to μ as $n \to \infty$.

$\overline{X}_n \to \mu$ AS $n \to \infty$

The law of large numbers can be used to justify the fact that if we are given a large number of independent trials $X_1, \ldots, X_n$ of the random variable X, then

$$\mathbb{E}[X] \approx \frac{1}{n}(X_1 + X_2 + \cdots + X_n).$$

To measure how reliable is the approximation of $\mathbb{E}[X]$ given by the sample average, consider the standard deviation of the trials $X_1, \ldots, X_n$:

$$s_X = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(\overline{X} - X_i)^2}.$$

$\overline{X} = \frac{X_1 + \cdots + X_n}{n} = O(1)$

$O(1)$, $O(n) \Rightarrow \ldots = O(1)$

A simple application of the Central Limit Theorem proves that the random variable

$\mathbb{E}[X] = \mu$

$$Y_n = \frac{\mu - \overline{X}}{s_X/\sqrt{n}} \to Y \in N(0,1)$$

$F_{Y_n}(x) \xrightarrow[n\to\infty]{} \Phi(x)$

converges in distribution to a standard normal random variable. We use this

IF n IS SUFFICIENTLY LARGE $\mathbb{P}(a < Y_n < b) \approx \frac{1}{\sqrt{2\pi}}\int_a^b e^{-\frac{1}{2}x^2}dx$

result to show that the true value μ of $\mathbb{E}[X]$ has about 95% probability to be in the interval

$$\mu \in [\overline{X} - 1.96\frac{s}{\sqrt{n}}, \overline{X} + 1.96\frac{s}{\sqrt{n}}].$$

Indeed, for n large

$$\mathbb{P}\left(-1.96 \leq \frac{\mu - \overline{X}}{s_X/\sqrt{n}} \leq 1.96\right) \approx \int_{-1.96}^{1.96} e^{-x^2/2}\frac{dx}{\sqrt{2\pi}} \approx 0.95.$$

Application to the Asian option

Consider now a European derivative with pay-off Y at maturity T. We approximate the price at time $t = 0$ by

$$\Pi_Y(0) = e^{-rT}\mathbb{E}_q[Y] \approx e^{-rT}\frac{Y_1 + \cdots + Y_n}{n}$$

where $Y_1, \ldots, Y_n$ is a large number of independent pay-off trials.

As the pay-off depends on the path of the stock price, the trials $Y_1, \ldots, Y_n$ can be created by first generating a sample of paths for the stock price.

Letting $0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of the interval $[0, T]$ with size $t_i - t_{i-1} = h$, we may construct a sample of n paths of the geometric Brownian motion on the given partition with the following simple Matlab function:

```
function Path=StockPath(s,sigma,r,T,N,n)
h=T/N;
W=randn(n,N);
q=ones(n,N);
Path=s*exp((r-sigma^2/2)*h.*cumsum(q')+sigma*sqrt(h)*cumsum(W'));
Path=[s*ones(1,n);Path];
```

Note carefully that the stock price is modeled as a geometric Brownian motion with mean of log return $\alpha = r - \sigma^2/2$, which means that the geometric Brownian motion is defined in the risk-neutral probability.

This is of course correct, since the expectation that we want to compute is in the risk-neutral probability measure.

In the case of the Asian call option with strike K and maturity T the pay-off is given by

$$Y = \left(\frac{1}{T} \int_0^T S(t)\,dt - K \right)_+ \approx \left(\frac{1}{N} \sum_{i=1}^{N} S(t_i) - K \right)_+ .$$

The following function computes the approximate price of the Asian option using the Monte Carlo method:

```
function [price, err]=MonteCarlo_AC(s,sigma,r,K,T,N,n)
tic
stockPath=StockPath(s,sigma,r,T,N,n);
payOff=max(0,mean(stockPath)-K);
price=exp(-r*T)*mean(payOff);
err=1.96*std(payOff)/sqrt(n);
toc
```

The function also returns the error in the 95% confidence interval, that is

$$\text{Err} = 1.96 \frac{s}{\sqrt{n}}.$$

For example, by running the command

```
[price, err]=MonteCarlo_AC(100,0.5,0.05,100,1/2,100,1000000)
```

we get `price`=8.5799, `err`=0.0283, which means that the Black-Scholes price of the Asian option with the given parameters has 95% probability to be in the interval 8.5799 ± 0.0283. The calculation took about 4 seconds. Note that the 95% confidence is $0.0565/8.5799 * 100 \approx 0.66\%$ of the price.

Iin order to reduce the error, i.e., to shrink the confidence interval, of the Monte Carlo approximation, one needs to either

(i) increase the number of trials n or

(ii) reduce the standard derivation s. Increasing n can be very costly in terms of computational time, hence the approach (ii) is more efficient.

There exist several methods to decrease the standard deviation of a Monte Carlo computation, which are known as **variance reduction techniques**. An example for the Asian option can be found in the book

BREAK UNTIL 16.15

Solution to Exercise 6.23

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$$\Pi_{AC}(0) = e^{-rT}\,\mathbb{E}_q[Y_{CALL}] \qquad Y_{CALL} = \left(\frac{1}{T}\int_0^T S(t)\,dt - K\right)_+$$

$$\Pi_{AP}(0) = e^{-rT}\,\mathbb{E}_q[Y_{PUT}] \qquad Y_{PUT} = \left(K - \underbrace{\frac{1}{T}\int_0^T S(t)\,dt}_{z}\right)_+$$

$$\Pi_{AC}(0) - \Pi_{AP}(0) = e^{-rT}\,\mathbb{E}_q[Y_{CALL} - Y_{PUT}]$$

$$= \{ (z-K)_+ - (K-z)_+ = z - K \text{ FOR ALL } z \}$$

$$= e^{-rT}\,\mathbb{E}_q\left[\frac{1}{T}\int_0^T S(t)\,dt - K\right]$$

$$= e^{-rT}\,\mathbb{E}_q\left[\frac{1}{T}\int_0^T S(t)\,dt\right] - K e^{-rT}$$

$$= e^{-rT}\,\frac{1}{T}\int_0^T \mathbb{E}_q[S(t)]\,dt - K e^{-rT}$$

$$= e^{-rT}\,\frac{1}{T}\int_0^T \mathbb{E}_q[\underbrace{e^{-rt}S(t)}_{S^*(t)}]\,e^{rt}\,dt - K e^{-rT}$$

$$\left(\mathbb{E}_q[S^*(t)] = \mathbb{E}_q[S^*(0)] = S(0)\right)$$

$$= e^{-rT}\,\frac{S_0}{T}\int_0^T e^{rt}\,dt - K e^{-rT}$$

$$= e^{-rT}\,\frac{S_0}{T}\,\frac{e^{rT}-1}{r} - K e^{-rT}$$

$$= e^{-rT}\left[S_0\,\frac{e^{rT}-1}{rT} - K\right]$$

Solution to Exercise 6.24

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$$\Pi^{(G)}_{AC}(0) = e^{-rT} E_q[Q]$$

$$Q = \left(e^{\frac{1}{T}\int_0^T \log S(t)\,dt} - K\right)_+$$

$$S(t) = S_0 e^{\alpha t + \sigma W(t)}$$ IN THE PHYS. PROBABILITY $\mathbb{P}$

$$= S_0 e^{(r-\frac{\sigma^2}{2})t + \sigma W^{(q)}(t)}$$ IN THE RISK-NEUTRAL (OR MARTINGALE) PROBABILITY $\mathbb{P}_q$

$\log S_0 + \log e^{(\dots)}$

$$\log S(t) = \log S_0 + \left(r - \frac{\sigma^2}{2}\right)t + \sigma W^{(q)}(t)$$

$$e^{\frac{1}{T}\int_0^T \log S(t)\,dt} = e^{\frac{1}{T}\int_0^T [\log S_0 + (r-\frac{\sigma^2}{2})t + \sigma W^{(q)}(t)]\,dt}$$

$\log S(t)$

$$= e^{\log S_0 + (r-\frac{\sigma^2}{2})\frac{T}{2} + \frac{\sigma}{T}\int_0^T W^{(q)}(t)\,dt}$$

$$= S_0 e^{(r-\frac{\sigma^2}{2})\frac{T}{2}} e^{\frac{\sigma}{T}\int_0^T W^{(q)}(t)\,dt}$$

$$Q = \left(S_0 e^{(r-\frac{\sigma^2}{2})\frac{T}{2}} e^{\frac{\sigma}{T}\int_0^T W^{(q)}(t)\,dt} - K\right)_+$$

WHAT IS THE DENSITY OF THIS R.V.?

THEOREM 6.6: FOR ALL $g : (0, \infty) \to \mathbb{R}$

$$X(T) = g(T)W(T) - \int_0^T g'(s)W(s)\,ds \in N(0, \Delta(t))$$

$$\Delta(t) = \int_0^t g(s)^2\,ds$$

WE WANT THIS THEOREM TO CHOOSE g IN TO DERIVE THE DENSITY OF $\int_0^T W(t)\,dt$. WE NEED

$$g'(t) = -1 \qquad g(T) = 0 \Rightarrow g(t) = T - t$$

HENCE
$$\int_0^T W(t)\,dt \in N(0, \Delta(T)) = N\left(0, \frac{T^3}{3}\right)$$

WHERE
$$\Delta(T) = \int_0^T (T-t)^2\,dt = T^3/3$$

$$Q = \left(S_0 e^{\frac{1}{2}\left(r - \frac{\sigma^2}{2}\right)T + \frac{\sigma}{T} G} - K\right)_+$$

$$\frac{\sigma}{T}\sqrt{\frac{T^3}{3}}\,\frac{G}{\sqrt{\frac{T^3}{3}}} = \frac{\sigma}{T}\sqrt{\frac{T^3}{3}}\,H$$

$H \in N(0,1)$

WHERE $G \in N\left(0, T^3/3\right)$

FOR THE STANDARD CALL

$$Y = \left(S_0 e^{(r - \sigma^2/2)T + \sigma\sqrt{T}H} - K\right)_+$$

$$\sigma W(T) = \sigma\sqrt{T}\,\frac{W(T)}{\sqrt{T}}$$ (H)

$H \in N(0,1)$

$$\Theta = \left(S_0 e^{\frac{1}{2}\left(r - \frac{\sigma^2}{2}\right)T + \sigma \sqrt{\frac{T}{3}} H} - K \right)_+$$

HENCE THE PRICE CAN BE COMPUTED AS FOR THE STANDARD CALL