# Lecture_24

den 11 december 2020 12:12

PDF Lecture_24

Options and Mathematics: Lecture 24

December 11, 2020

## Standard European derivatives on a dividend-paying stock

In this llecture we compute the Black-Scholes price of standard European derivatives on a dividend-paying stock.

In a frictionless market this means that the price of the stock drops at some time $t_0 \in (0, T)$ of a quantity $D < S(t_0)$, which is deposited in the account of the shareholders.

Letting $S(t_0^-) = \lim_{t \to t_0^-} S(t)$, we then have

$$S(t_0) = S(t_0^-) - D.$$

We assume that on each of the intervals $[0, t_0)$, $[t_0, T]$, the stock price follows a geometric Brownian motion.

In the risk-neutral probability $\mathbb{P}_q$ this means that

$$S(s) = S(t)e^{(r-\frac{1}{2}\sigma^2)(s-t)+\sigma(W^{(q)}(s)-W^{(q)}(t))}, \quad t \in [0, t_0),\ s \in [t, t_0)$$
$$S(s) = S(u)e^{(r-\frac{1}{2}\sigma^2)(s-u)+\sigma(W^{(q)}(s)-W^{(q)}(u))}, \quad u \in [t_0, T],\ s \in [u, T].$$

BEFORE THE DIVIDEND IS PAYED

AFTER THE DIVIDEND IS PAYED

$D = a S(t_0^-)$ , $a \in (0,1)$

$\Rightarrow S(t_0) = S(t_0^-) - D = (1-a) S(t_0^-) > 0$

To simplify the discussion we consider the case in which the dividend $D$ is expressed as percentage of the stock price just before the dividend is paid, i.e., $D = aS(t_0^-)$, for some $a \in (0,1)$.

Note that this means that we do not know at time $t = 0$ what amount the dividend will pay at time $t_0 > 0$, but we known for sure that $D < S(t_0^-)$.

**Theorem 6.18**

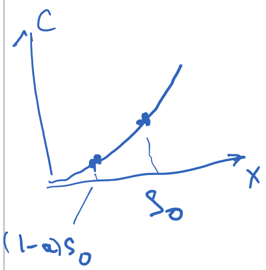
Consider the standard European derivative with pay-off $Y = g(S(T))$ and maturity $T$. Let $\Pi_Y^{(a,t_0)}(0)$ be the Black-Scholes price of the derivative at time $t = 0$ assuming that the underlying stock pays the dividend $aS(t_0^-)$ at time $t_0 \in (0,T)$, where $a \in (0,1)$. Then

$C^{(a,t_0)}(0, S_0, K, T) = C(0, (1-a)S_0, K, T)$

$C(0, x, K, T)$ is INCREASING IN $x$

$$\boxed{\Pi_Y^{(a,t_0)}(0) = v_0((1-a)S_0)}$$

where $v_0(x)$ is the pricing function in the absence of dividends.




**Exercise 6.28**[?]

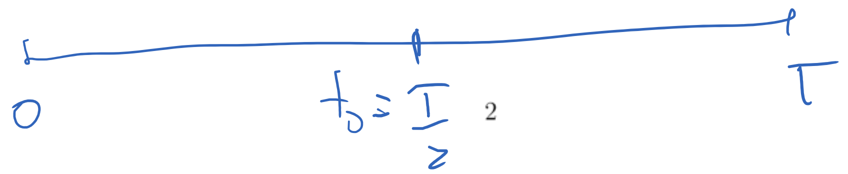
Use the previous result to show that the call option is less valuable at time $t = 0$ if the stock pays a dividend at time $t_0 > 0$. Give an intuitive explanation for this property.

**Exercise 6.29**

CALL WITH STRIKE $K = S(0)$

A standard European derivative pays the amount $Y = (S(T) - S(0))_+$ at time of maturity $T$. Find the Black-Scholes price $\cancel{\Pi_Y}(0)$ of this derivative at time $t = 0$ assuming $r > 0$ and that the underlying stock pays the dividend $(1 - e^{-rT})S(\frac{T}{2}-)$ at time $t = \frac{T}{2}$. Compute the probability of positive return for a constant portfolio which is short 1 share of the derivative and short $S(0)e^{-rT}$ shares of the risk-free asset (assume $B(0) = 1$).

$a = (1 - e^{-rT}) \in (0,1)$

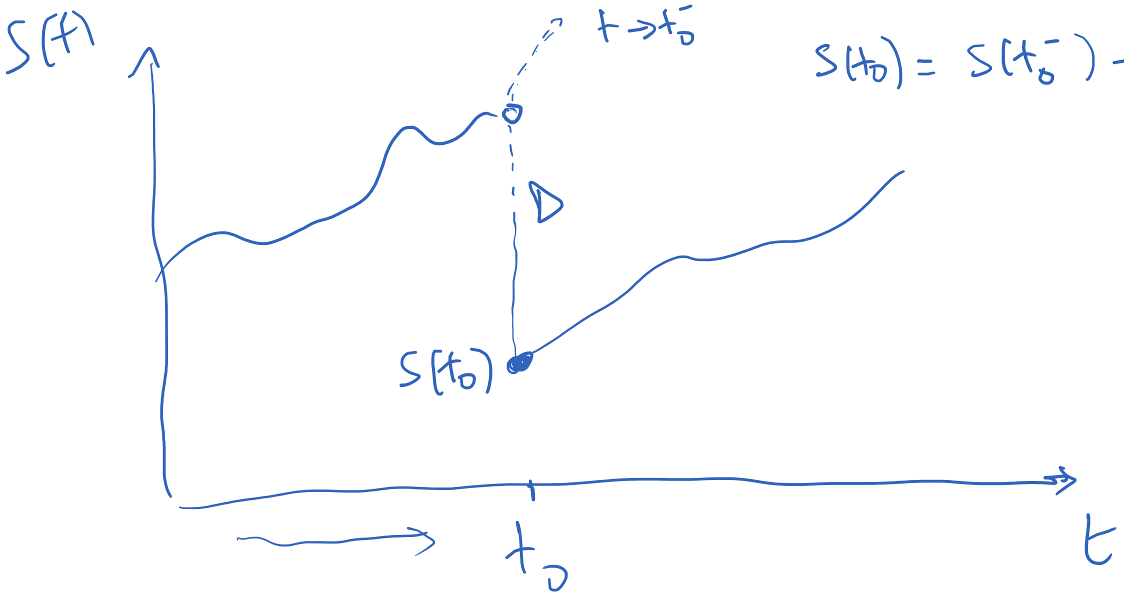


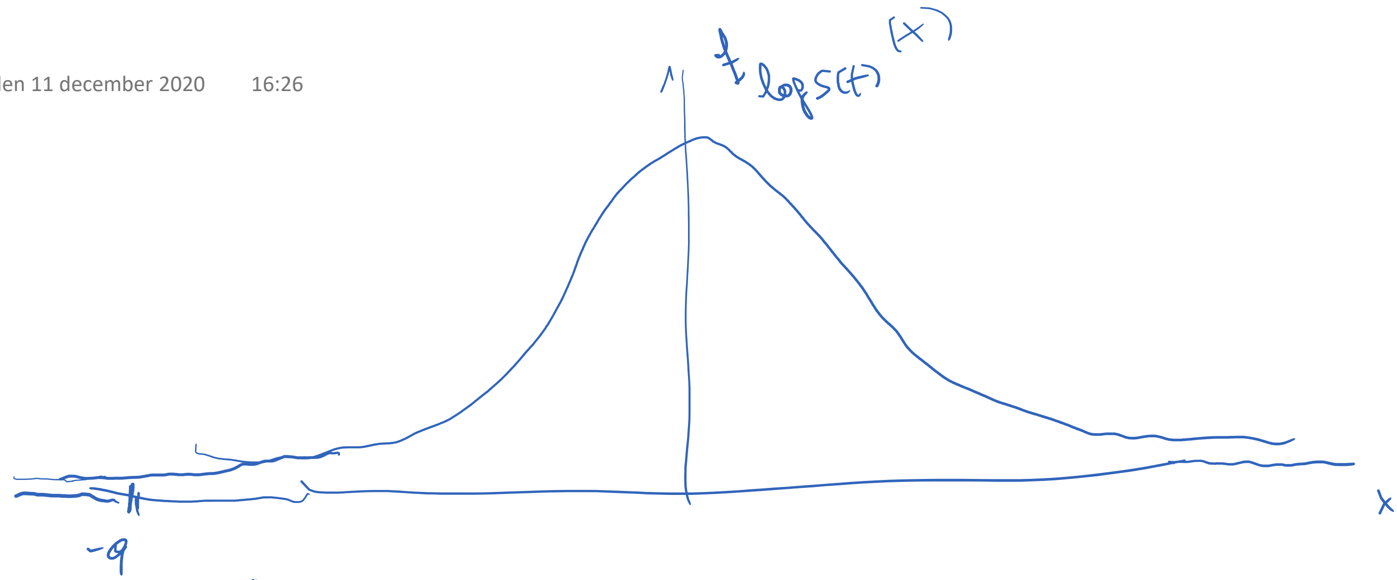

2

den 11 december 2020 16:19

$$\lim_{t \to t_0^-} S(t) = S(t_0^-)$$

$$S(t_0) = S(t_0^-) - D$$






$\mathbb{P}(\log S(t) < -9) > 0$ (VERY SMALL BUT POSITIVE)

$\Rightarrow \mathbb{P}(S(t) < 10^{-9}) > 0$

IN THE N-PERIOD BINOMIAL MODEL,

$S(t) > \underbrace{S(0)e^{Nd}}$

IF $D < S(0)e^{Nd}$ THEN

$S(t) - D > 0$ FOR ALL $t = 1, \dots, N$

$\frac{D}{S(0)} < e^{Nd}$

$\to \approx 1\% = 0.01$

# Solution to Exercise 6.29

den 11 december 2020 16:41

$$v_0(x) = e^{-rT} \int_{\mathbb{R}} g\left(\underbrace{x}_{\to (1-a)S(0)} e^{(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}y}\right) e^{-\frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}}$$

$$g(z) = (z - S(0))_+ \qquad a = (1 - e^{-rT})$$

$$1 - a = e^{-rT}$$

$$\Pi(0) = v_0((1-a)S(0)) =$$

$$= e^{-rT} \int_{\mathbb{R}} \left((1-a)S(0) e^{(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}Y} - S(0)\right)_+ e^{-\frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}}$$

$$= S(0) e^{-rT} \int_{\mathbb{R}} \left(e^{-\frac{\sigma^2}{2}T + \sigma\sqrt{T}Y} - 1\right)_+ e^{-\frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}}$$

$$\left\{ e^{-\frac{\sigma^2}{2}T + \sigma\sqrt{T}Y} - 1 > 0 \iff -\frac{\sigma^2}{2}T + \sigma\sqrt{T}Y > 0 \right.$$

$$\left. \iff Y > \frac{\frac{\sigma^2}{2}T}{\sigma\sqrt{T}} = \frac{\sigma\sqrt{T}}{2} \right\}$$

$$= S(0) e^{-rT} \int_{\frac{\sigma\sqrt{T}}{2}}^{\infty} \left(e^{-\frac{\sigma^2}{2}T + \sigma\sqrt{T}Y} - 1\right) e^{-\frac{1}{2}Y^2} \frac{dy}{\sqrt{2\pi}}$$

$$= S(0) e^{-rT} \int^{\infty} e^{-\frac{\sigma^2}{2}T + \sigma\sqrt{T}Y - \frac{1}{2}Y^2} \frac{dy}{}$$

$$= S(0)e^{-rT}\int_{\frac{\sigma\sqrt{T}}{2}} e^{\underbrace{-\frac{1}{2}y^2+\sigma\sqrt{T}y-\frac{1}{2}\sigma^2 T}_{-\frac{1}{2}(\underbrace{y-\sigma\sqrt{T}}_{z})^2}}\frac{dy}{\sqrt{2\pi}}$$

$$- S(0)e^{-rT}\int_{\frac{\sigma\sqrt{T}}{2}}^{\infty} e^{-\frac{1}{2}y^2}\frac{dy}{\sqrt{2\pi}}$$

$$= S(0)e^{-rT}\int_{-\frac{\sigma\sqrt{T}}{2}}^{\infty} e^{-\frac{1}{2}z^2}\frac{dz}{\sqrt{2\pi}}$$

$$- S(0)e^{-rT}\int_{-\infty}^{-\frac{\sigma\sqrt{T}}{2}} e^{-\frac{1}{2}y^2}\frac{dy}{\sqrt{2\pi}}$$

$$= S(0)e^{-rT}\left[\Phi\left(\frac{\sigma\sqrt{T}}{2}\right)-\Phi\left(-\frac{\sigma\sqrt{T}}{2}\right)\right]$$

$$\left\{\Phi(z)+\Phi(-z)=1 \Rightarrow \Phi(-z)=1-\Phi(z)\right\}$$

$$= \underbrace{S(0)e^{-rT}\left[2\Phi\left(\frac{\sigma\sqrt{T}}{2}\right)-1\right]}$$

FIRST PART

SECOND PART

$$V(t) = -\Pi(t) - S(0)e^{-rT}\underbrace{B(t)}^{B(0)e^{rt}=e^{rt}}$$

$$= -\Pi(t) - S(0)\, e^{-r(T-t)}$$

$$R = V(T) - V(0) =$$

$$= -\underbrace{\Pi(T)}_{Y} - S(0) - \left(-\Pi(0) - S(0)e^{-rT}\right)$$

$$= -(S(T)-S(0))_+ - S(0)$$

$$+ \Pi(0) + S(0)e^{-rT}$$

$$= -(S(T)-S(0))_+ - S(0) + \underline{S(0)e^{-rT}}$$

$$+ \underline{S(0)}e^{-rT}\left(2\Phi\left(\frac{\sigma\sqrt{T}}{2}\right) - 1\right)$$

$$= -(S(T)-S(0))_+ - S(0) + S(0)e^{-rT}\, 2\Phi\left(\frac{\sigma\sqrt{T}}{2}\right)$$

$$\begin{cases} -S(T) + S(0)e^{-rT}\, 2\Phi\left(\frac{\sigma\sqrt{T}}{2}\right) & \text{if } S(T) > S(0) \\ \underbrace{-S(0) + S(0)e^{-rT}\, 2\Phi\left(\frac{\sigma\sqrt{T}}{2}\right)}_{} & \text{if } \underline{S(T) < S(0)} \end{cases}$$

$$S(0)\left[e^{-rT}\, 2\Phi\left(\frac{\sigma\sqrt{T}}{2}\right) - 1\right]$$

$$\vdots$$

LOOK THE REST IN THE BOOK