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Options and Mathematics: Lecture 26

December 16, 2020

Review of the first part of the course

In the first part of the course (Chapter 1) we study some qualitative properties of options prices based on the arbitrage free principle. One of these properties is the put call parity:

Theorem 1.2(v)

$$\boxed{C(t, S(t), K, T) - P(t, S(t), K, T) = S(t) - KB(t, T)}$$

$S(t) = \Pi^{M}(t)$

$K = \mathrm{For}_M(t, T)$

Remember that in the exam you have to provide and explain all steps of the proof (including those missing in the book)

In Part I we also introduced the fundamental concept of optimal exercise time for American put options:

Definition 1.2. A time $t < T$ is called an **optimal exercise time** for the American put with value $\widehat{P}(t, S(t), K, T)$ if $S(t) < K$ and $\widehat{P}(t, S(t), K, T) = (K - S(T))_+$.

Recall that in the exam you have to explain the meaning of all mathematical symbols as well as the financial interpretation of the definition.

Exercises

The typical exercise in Part I consists in replicating the value of a derivative with continuous, piecewise linear pay-off using European call and put options. To this purpose the following result is used.

PARTICULAR CASE OF THEOREM 1.1 (WHEN $C_A \equiv 0$)

If $\mathcal{A}$ is not an arbitrage portfolio and it is known at time t that $V_{\mathcal{A}}(T) = 0$, then $V_{\mathcal{A}}(t) = 0$

Exercise 1.10 (Solution in the book)

A forward contract on an asset $\mathcal{U}$ stipulated at time t and with maturity $T > t$ is a costless agreement between two parties in which one party promises to sell, and the other one to buy, the asset $\mathcal{U}$ at time T for a given price $\mathrm{For}_{\mathcal{U}}(t, T)$, which is called forward price of the asset.

IF $\Pi^u(T) < For_u(t,T)$ THEN THE BUYER PARTY WILL INCUR IN A LOSS.

Assume that the asset $\mathcal{U}$ does not pay dividends in the interval (t, T). Show that the forward price of $\mathcal{U}$ in an arbitrage-free market is given by

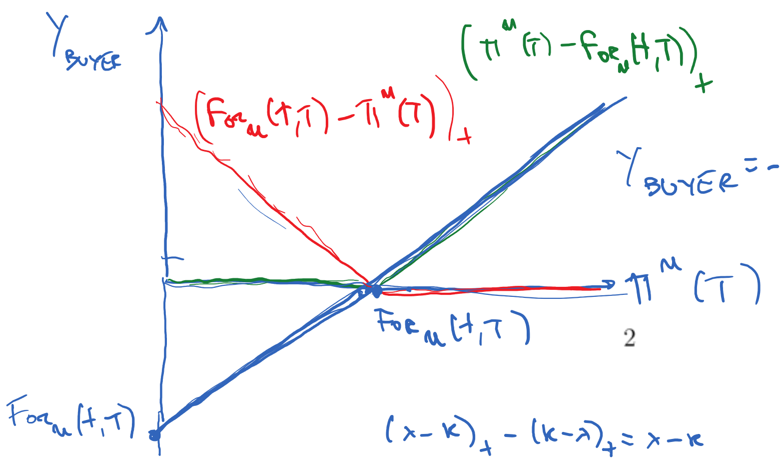
$$\mathrm{For}_{\mathcal{U}}(t, T) = \frac{\Pi^{\mathcal{U}}(t)}{B(t, T)}.$$

SPOT PRICE

Derive the analogous formula when $\mathcal{U}$ pays the dividend D at time $t_0 \in (t, T)$.

"PAY-OFF" FOR THE BUYER PARTY $= \Pi^u(T) - For_u(t,T) \leftarrow Y_{BUYER}$

" "PAY-OFF" " " SELLER PARTY $= For_u(t,T) - \Pi^u(T)$



PAY-OFF OF A PUT WITH STRIKE $For_u(t,T)$

$$Y_{BUYER} = -(For_u(t,T) - \Pi^u(T))_+ + (\Pi^u(T) - For_u(t,T))_+$$

PAY-OFF OF A CALL WITH STRIKE $For_u(t,T)$

2

$(x-k)_+ - (k-x)_+ = x-k$

$V(t) \equiv$ VALUE OF FORWARD CONTRACT AT TIME t =

$C(t, \Pi^u(t), For_u(t,T), T) - P(t, \Pi^u(t), For_u(t,T), T) = \Pi^u(t) - For_u(t,T)\, B(t,T)$ ← PUT-CALL PARITY

SINCE FORWARD HAVE ZERO PRICE, THEN

$V(t) \equiv 0$ MUST HOLD $\Rightarrow For_u(t,T) = \dfrac{\Pi^u(t)}{B(t,T)}$

V(T) = 0 MUST HOLD $\Rightarrow$ $For_M(t,T) = \dfrac{\Pi^u(t)}{B(t,T)}$

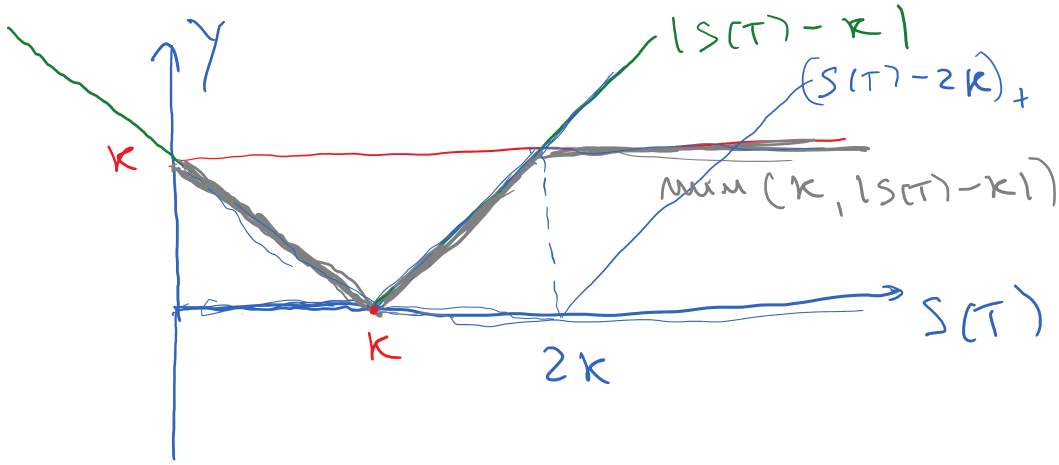
EXAMPLE: IF $r \equiv$ CONSTANT, THEN

$B(t,T) = e^{-r(T-t)}$, HENCE $For_M(t,T) = \Pi^u(t)\, e^{r(T-t)}$

Exercise 1.13 (Answer in the book)

Let $K, T > 0$ and consider the European style derivative with pay-off $Y = \min(K, |S(T) - K|)$ at maturity T, where $S(t)$ is the price of the underlying stock at time t. Write the value of this derivative in terms of the value of call and put options.

SOLUTION: ALWAYS START BY DRAWING THE PAY-OFF FUNCTION.

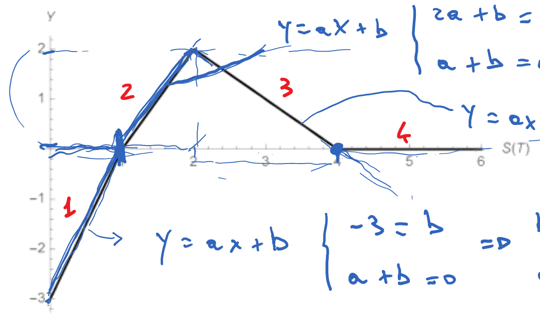


$$Y = (K - S(T))_+ + (S(T) - K)_+ - (S(T) - 2K)_+$$

$$\Pi^u(t) = P(t, S(t), K, T) + C(t, S(t), K, T) - C(t, S(t), 2K, T)$$

Exercise 1.15

Find a constant portfolio consisting of European puts that replicates the European derivative with maturity T and pay-off Y depicted in Figure 1.



$y = ax + b \quad \begin{cases} 2a + b = 2 \\ a + b = 0 \end{cases} \Rightarrow \begin{cases} a = 2 \\ b = -2 \end{cases}$

$y = ax + b \quad \begin{cases} 4a + b = 0 \\ 2a + b = 2 \end{cases} \Rightarrow \begin{matrix} a = -1 \\ b = 4 \end{matrix}$

$y = ax + b \quad \begin{cases} -3 = b \\ a + b = 0 \end{cases} \Rightarrow \begin{matrix} b = -3 \\ a = 3 \end{matrix}$

Figure 1: Remark: For $S(T) > 4$ the pay-off is identically zero

1 THE FIRST LINE IS $3S(T) - 3 = 3(S(T)-1) = -3(1-S(T))_+$

2 THE SECOND LINE IS $2S(T) - 2 = 2(S(T)-1)_+$

3 THE THIRD LINE IS $(-S(T) + 4)$, WHICH IS OBTAINED BY SUBTRACTING $3(S(T)-2)_+$ FROM THE 2^{nd} LINE: $2(S(T)-1)_+ - 3(S(T)-2)_+ = -S(t) + 4$, FOR $S(T) > 4$

4 TO REPRODUCE THE FOURTH LINE WE HAVE TO ADD $(S(T)-4)_+$

HENCE

$$Y = -3(1-S(T))_+ + 2(S(T)-1)_+ - 3(S(T)-2)_+ + (S(T)-4)_+$$

$$\Rightarrow \Pi^M(t) = -3P(t,S(t),1,T) + 2C(t,S(t),1,T) - 2C(t,S(t),2,T) + C(t,S(t),4,T)$$

IF WE WANT TO USE ONLY PUT OPTIONS, AS ASKED IN THE EXERCISE, WE USE $(S(T)-K)_+ = (K - S(T))_+ + S(T) - K$, HENCE

$$Y = -3(1-S(T))_+ + 2(1-S(T))_+ + 2(S(T)-1) - 3(2-S(T))_+ - 3(S(T)-2) + (4-S(T))_+ + S(T) - 4$$

THE RED TERMS CANCEL OUT

Exercise 1.19 Let $r \geq 0$ and assume that the stock pays the dividend D at time t_0, where $t < t_0 < T$. Show that if

$$D > Ke^{rt_0}(1 - e^{-rT}),$$

then it is not optimal to exercise the American put with strike K and maturity T in the interval $[t, t_0)$. HINT: Use the put call parity with dividends

$$C(t, S(t), K, T) - P(t, S(t), K, T) = S(t) - Ke^{-r(T-t)} - De^{-r(t_0-t)}, \quad t < t_0, \quad (*)$$

to show that the value of the put is larger than its pay-off before the dividend is payed.

SOLUTION: SINCE THE AMERICAN PUT IS MORE VALUABLE THAN THE EUROPEAN PUT, THEN

$$\hat{P}(t,S(t),K,T) \geq P(t,S(t),K,T) = C(t,S(t),K,T) - S(t) + Ke^{-r(T-t)} + De^{-r(t_0-t)}, \quad t < t_0$$

WHERE WE USED (*). SINCE $C \geq 0$, THEN

$$P(t,S(t),K,T) \geq Ke^{-r(T-t)} + De^{-r(t_0-t)} - S(t)$$

THE RIGHT HAND SIDE IS LARGER THAN $K - S(t)$ WHEN $Ke^{-r(T-t)} + De^{-r(t_0-t)} > K$, THAT IS

$$D > K(1 - e^{-r(T-t)})\, e^{r(t_0-t)} = Ke^{rt_0}(e^{-rt} - e^{-rT})$$

SINCE $e^{-rt} < 1$, THIS IS TRUE IN PARTICULAR WHEN $D > Ke^{rt_0}(1 - e^{-rT})$. THUS IF THE DIVIDEND D SATISFIES THIS LOWER BOUND, AND THE AMERICAN PUT IS IN THE MONEY (I.E., $S(t) < K$), THEN

$$\hat{P}(t,S(t),K,T) > (K - S(t)) = (K - S(t))_+$$

HENCE t IS NOT AN OPTIMAL EXERCISE TIME

Exercise 1.25 (Solution in the book)

A European derivative on a stock pays the amount

$$Y = (\min(S(T) - 10, 20 - S(T), 2))_+ - 1$$

at maturity T. Draw the pay-off function of the derivative and find a constant portfolio of European call/put options that replicates the derivative.

Exercise 1.26

Find a constant portfolio consisting of European calls and puts with expiration date T such that the value of the portfolio at time T equals

$$V(T) = \min[(S(T) - K)_+, (L - S(T))_+, (L - K)/4],$$

where $0 < K < L$. HINT: Draw the graph of the pay-off function and write it as a combination of call and put pay-offs.

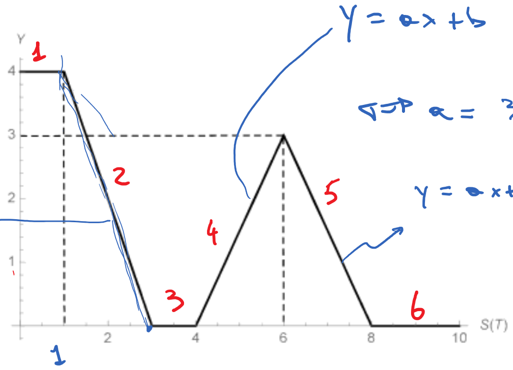
Exercise 1.27 (Answer in the book)

Find a constant portfolio consisting of European calls and/or puts that replicates the European derivative with maturity T and pay-off Y depicted in Figure 2.

$Y = ax + b \quad \begin{cases} 4a + b = 0 \\ 6a + b = 3 \end{cases}$

$\Longrightarrow a = 3/2, \; b = -6$

$Y = ax + b \quad \begin{cases} 3a + b = 0 \\ a + b = 4 \end{cases} \Longrightarrow a = -2, \; b = 6$



$Y = ax + b \quad \begin{cases} 6a + b = 3 \\ 8a + b = 0 \end{cases} \Longrightarrow a = -3/2, \; b = 12$

Figure 2: Remark: For $S(T) > 10$ the pay-off is identically zero. Careful with the angles!

Solution to Exercise 1.26

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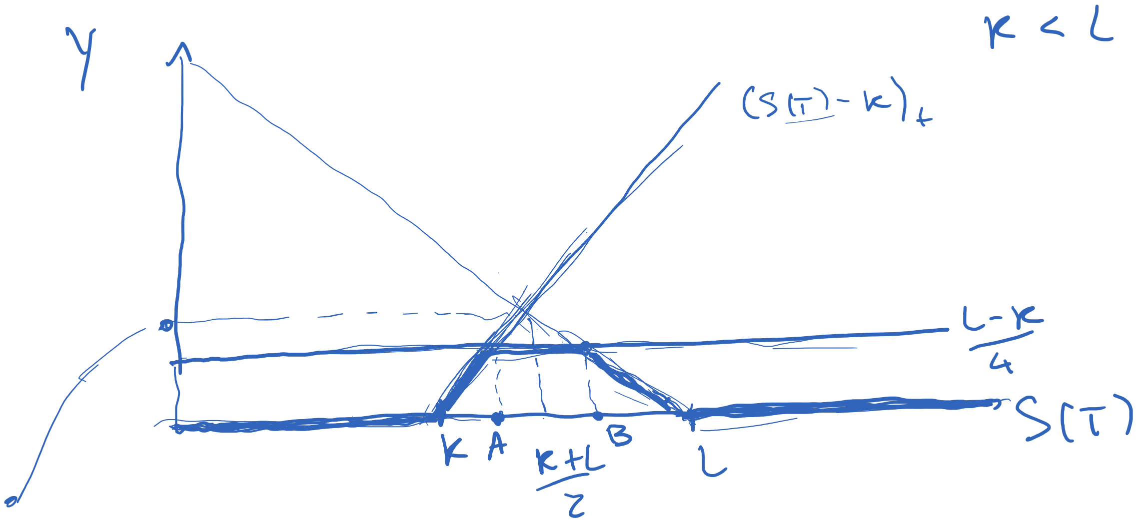
$$Y = \min\left((S(T)-K)_+ , (L-S(T))_+ , \frac{L-K}{4} \right)$$

$K < L$



$$\frac{K+L}{2} - K = \frac{L-K}{2} > \frac{L-K}{4}$$

$$Y = (S(T)-K)_+ - (S(T)-A)_+ - (S(T)-B)_+ + (S(T)-L)_+$$

A is the stock price $S(T)$:

$$S(T) - K = \frac{L-K}{4} \Rightarrow S(T) = \frac{L}{4} + \frac{3K}{4} = \frac{L+3K}{4}$$

B is the stock price $S(T)$:

$$L - S(T) = \frac{L-K}{4} \Rightarrow S(T) = \frac{3L+K}{4}$$

$$\Pi^M(t) = c(t, S(t), K, T) - c(t, S(t), A, T) - c(t, S(t), B, T) + c(t, S(t), L, T)$$

Solution to Exercise 1.27

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THE SECOND LINE IS $(-2S(T)+6)$

$= 2(3-S(T))$

I CAN USE $2(3-S(T))_+$, BUT ONLY FOR $1 < S(T) < 3$, BECAUSE FOR $0 < S(T) < 1$ THE PAY-OFF IS CONSTANT $= 4$. USING THAT $4 = 2(3-S(T))_+ - 2(1-S(T))_+$ FOR $S(T) < 1$, THEN THE LINES 1, 2 AND 3 CAN BE WRITTEN AS $2(3-S(T))_+ - 2(1-S(T))_+$

THE LINE 4 IS $\frac{3}{2}S(T) - 6 = \frac{3}{2}(S(T)-4)$

$= \frac{3}{2}(S(T)-4)_+$ FOR $S(T) > 4$

THE LINE 5 IS $-\frac{3}{2}S(T)+12$. TO OBTAIN THIS LINE FROM LINE 4 WE USE

$$-\frac{3}{2}S(T)+12 = \frac{3}{2}S(T) - 6 - 3S(T) + 18$$

$$= \frac{3}{2}S(T) - 6 - 3(S(T)-6)$$

$$= \frac{3}{2}S(T) - 6 - 3(S(T)-6)_+$$

FOR $S(T) > 6$

HENCE TO OBTAIN LINE 5 WE HAVE TO ADD $-3(S(T)-6)_+$.

FINALLY, LINE 6 $(Y = 0)$ IS OBTAINED FROM LINE 5 AS $-\frac{3}{2}S(T)+12 + \frac{3}{2}S(T) - 12$

$$= -\frac{3}{2}S(T) + 12 + \frac{3}{2}(S(T) - 8)$$

$$= -\frac{3}{2}S(T) + 12 + \frac{3}{2}(S(T) - 8)_+$$

FOR $S(T) > 8$

CONCLUSION: THE PAY-OFF Y IS OBTAINED BY ADDING UP THE RED TERMS ABOVE:

$$Y = 2(3 - S(T))_+ - 2(1 - S(T))_+ + \frac{3}{2}(S(T) - 4)_+$$
$$-3(S(T) - 6)_+ + \frac{3}{2}(S(T) - 8)_+$$

HENCE:

$$\pi_Y(t) = 2P(t, S(t), 3, T) - 2P(t, S(t), 1, T)$$
$$+ \frac{3}{2}C(t, S(t), 4, T) - 3C(t, S(t), 6, T)$$
$$+ \frac{3}{2}C(t, S(t), 8, T)$$