Lecture_27

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Lecture_27

Options and Mathematics: Lecture 26

December 16, 2020

Review of the second part of the course

In the second part of the course (Chapter 2,3,4) we introduced the Binomial Model to price European and American derivatives using binomial trees. This formulation is useful for the numerical implementation of the model (see the projects). The following is a summary of important formulas introduced in Part II.

Binomial stock price

$$
\begin{tikzcd}[row sep=small]
 & S(1) \arrow[r] \arrow[dr] & S(2) & \cdots \\
S(0) \arrow[ur] \arrow[dr] & & S(2) & \\
 & S(1) \arrow[ur] \arrow[r] & S(2) &
\end{tikzcd}
$$

$$S(t_i) = \begin{cases} S(t_{i-1})e^u & \text{with probability } p_u = p \\ S(t_{i-1})e^d & \text{with probability } p_d = 1-p \end{cases},$$

Mean of log-return α and volatility σ of the binomial stock price (Definition 2.1)

$$\mathbb{E}[\log S(i) - \log S(i-1)] = \alpha = \frac{1}{h}[pu + (1-p)d], \quad \sigma = \frac{u-d}{\sqrt{h}}\sqrt{p(1-p)}. \quad \sigma^2 = \mathrm{VAR}[\log S(i) - \log S(i-1)]$$

Predictable portfolio process in the binomial market

$$h_S(t) = h_S(t, x_1, \dots, x_{t-1}), \quad h_B(t) = h_B(t, x_1, \dots, x_{t-1}), \quad x_i = u \text{ or } d$$

$h_S(t), h_B(t)$

$[t-1 \text{ --------- } t]$

Self-financing portfolio in the binomial market (Definition 2.5)

$$h_S(t)S(t-1) + h_B(t)B(t-1) = h_S(t-1)S(t-1) + h_B(t-1)B(t-1)$$

Recurrence formula for the value of self-financing portfolios in the binomial market (Theorem 2.1)

$$V(t) = e^{-r}[q_u V^u(t+1) + q_d V^d(t+1)], \quad \text{for } t = 0, \dots, N-1.$$

where

$$q_u = \frac{e^r - e^d}{e^u - e^d}, \quad q_d = 1 - q_u$$

Arbitrage portfolio in the binomial market (Definition 2.7)

A portfolio process $\{(h_S(t), h_B(t)\}_{t\in\mathcal{I}}$ invested in a binomial market is called an **arbitrage portfolio process** if it is predictable and if its value $V(t)$ satisfies

1) $V(0) = 0$;
2) $V(N, x) \geq 0$, for all $x \in \{u, d\}^N$;
3) There exists $y \in \{u, d\}^N$ such that $V(N, y) > 0$.

Theorem 2.4 (For the exam you only need to know the proof for the 1-period binomial model).

There exists a self-financing arbitrage portfolio in the binomial market if and only if $r \notin (d, u)$.

$d < r < u$ IS EQUIVALENT TO THE BINOMIAL MARKET BEING ARBITRAGE-FREE $\Rightarrow$

Equivalently $q_u, q_d \in (0, 1)$, hence (q_u, q_d) defines a probability (**martingale** or **risk-neutral** probability)

Hedging/replicating portfolio for the European derivative with pay-off Y and maturity T (Definition 3.1)

$$V(T) = Y \text{ (hedging)}, \quad \Pi_Y(t) = V(t) \text{ for all } t \leq T \text{ (replicating)}$$

PAY-OFF Y AND
MATURITY $T=N$

Binomial price of European derivatives (Definition 3.2)

$$\Pi_Y(t) := e^{-r(N-t)} \sum_{(x_{t+1},\dots,x_N)\in\{u,d\}^{N-t}} q_{x_{t+1}}\cdots q_{x_N} Y(x_1,\dots,x_N)$$

At $t=0$ (present time)

$$\boxed{\Pi_Y(0) = e^{-rN} \sum_{x\in\{u,d\}^N} (q_u)^{N_u(x)}(q_d)^{N_d(x)} Y(x)}$$

Recurrence formula for $\Pi_Y(t)$:

$$\Pi_Y(N) = Y, \quad \text{and} \quad \Pi_Y(t) = e^{-r}[q_u\Pi_Y^u(t+1)+q_d\Pi_Y^d(t+1)], \quad \text{for } t\in\{0,\dots,N-1\}.$$

Replicating portfolio of European derivatives in the binomial model (Theorem 3.3)

$$h_S(t) = \frac{1}{S(t-1)}\frac{\Pi_Y^u(t)-\Pi_Y^d(t)}{e^u-e^d},$$
$$h_B(t) = \frac{e^{-r}}{B(t-1)}\frac{e^u\Pi_Y^d(t)-e^d\Pi_Y^u(t)}{e^u-e^d}.$$

Recurrence formula for the binomial price of American derivatives

$$\widehat{\Pi}_Y(N) = Y(N)$$
$$\widehat{\Pi}_Y(t) = \max[Y(t), e^{-r}(q_u\widehat{\Pi}_Y^u(t+1) + q_d\widehat{\Pi}_Y^d(t+1))], \quad t\in\{0,\dots,N-1\}$$

Replicating portfolio of American derivatives and cash-flow generated by this portfolio

$$\left\{\begin{aligned}\widehat{h}_S(t) &= \frac{1}{S(t-1)}\frac{\widehat{\Pi}_Y^u(t)-\widehat{\Pi}_Y^d(t)}{e^u-e^d},\\ \widehat{h}_B(t) &= \frac{e^{-r}}{B(t-1)}\frac{e^u\widehat{\Pi}_Y^d(t)-e^d\widehat{\Pi}_Y^u(t)}{e^u-e^d}.\end{aligned}\right.$$

$$C(0)=0,\quad \underline{C(t)=\widehat{\Pi}_Y(t)-e^{-r}[q_u\widehat{\Pi}_Y^u(t+1)+q_d\widehat{\Pi}_Y^d(t+1)]},\quad t\in\{1,\dots,N-1\},$$

Exercises

The exercises in Part II consist above all in computing the binomial price and hedging portfolios of standard European/American derivatives, as well as the cash flow possibly generated by the replicating portfolio of American derivatives.

Exercise 4.6

THERE IS NO p

Consider a 3 period binomial model with parameters $u = \log(4/3)$, $d = \log(2/3)$, $r = \log(7/6)$, $S(0) = 27$. Consider a put option with strike $K = 24$ and expiring at time $T = 3$. Compute the binomial price of the put at time $t = 0$ under each of the following terms:

(i) The owner can exercise the derivative only at maturity (European style put option)

(ii) The owner can exercise at any time prior to or including maturity (American style put option)

(iii) The owner of the derivative is allowed to exercise at time $t = 2$ or at maturity, but *not* at time $t = 1$ (Bermuda style put option)

In cases (ii) and (iii), identify the optimal exercise times prior to expiration and the amount of cash that the seller can withdraw from the hedging portfolio if the buyer does not exercise the derivative optimally.

ASSUME $p = 1/2$

Addendum: Compute the expected return on each of the above derivative in the interval $[0, 3]$, where in case (ii),(iii) it is assumed that the the owner exercises the derivative at and only at the first optimal exercise time.

Exercise 3.4

A European derivative with expiration $T = N$ pays the amount $Y = \log(\frac{S(T)}{K})$. Find K such that the binomial price of the derivative at time $t = 0$ is zero. What is the financial meaning of this value of K? HINT: Use the identity

$$\binom{N}{k} k = N \binom{N-1}{k-1}.$$

ANSWER: $K = S(0)e^{N(q_u u + q_d d)}$.

Exercise 3.24

Derive the put-call parity satisfied at time $t = 0$ by the Asian call option with geometric average, which is the non-standard European style derivative with pay-off

$$Y(x) = \left(\left(\prod_{t=0}^{N} S(t) \right)^{1/(N+1)} - K \right)_+ .$$

Review of the third part of the course

The third part of the course (Chapters 5,6) started with the probabilistic reformulation of the Binomial options pricing model. We proved the following fundamental theorem.

Theorem 5.4 If $r \notin (d, u)$, there is no probability measure $\mathbb{P}_p$ on the sample space Ω_N such that the discounted stock price process $\{S^*(t)\}_{t=0,\dots,N}$ is a martingale. For $r \in (d, u)$, $\{S^*(t)\}_{t=0,\dots,N}$ is a martingale with respect to the probability measure $\mathbb{P}_p$ if and only if $p = q$, where

$$q = \frac{e^r - e^d}{e^u - e^d}.$$

and we have seen that the formula for the binomial brice at time $t = 0$ of the European derivative with pay-off Y and maturity T is equivalent to the risk-neutral pricing formula

$$\boxed{\Pi_Y(0) = e^{-rN}\mathbb{E}_q[Y].}$$

T = N

We have used this formula to define the Black-Scholes price of European derivative. In the Black-Scholes case the stock price is given by the Geometric Brownian motion

$$S(t) = S(0)e^{\alpha t + \sigma W(t)}$$

Definition 6.8 The Black-Scholes price at time $t = 0$ of the European derivative with pay-off Y at maturity T is given by the risk-neutral pricing formula

$$\boxed{\Pi_Y(0) = e^{-rT}\mathbb{E}_q[Y],}$$

i.e., it equals the expected value of the discounted pay-off in the risk-neutral probability measure of the Black-Scholes market.

In the case of standard European derivatives with pay-off $\boxed{Y = g(S(T))}$ the Black-Scholes price at time t can be written in the following integral form:

$$\Pi_Y(t) = v(t, S(t)), \quad \text{where} \quad v(t,x) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\mathbb{R}} g\left(xe^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}\, y}\right) e^{-\frac{y^2}{2}}\, dy, \quad \tau = T - t.$$

NUMBER OF STOCK SHARES IN THE REPLICATING PORTFOLIO

$$h_S(t) = \Delta(t, S(t)), \quad \Delta(t,x) = \partial_x v(t,x)$$

In the special case of the European call/put option we have proved the following theorem

Theorem 6.14 (for the exam you only need to know the derivation of the price of the call)

The Black-Scholes price at time t of the European call option with strike price $K > 0$ and maturity $T > 0$ is given by $C(t, S(t), K, T)$, where

$$v(t,x) = C(t, x, K, T) = x\Phi(d_{(+)}) - Ke^{-r\tau}\Phi(d_{(-)}), \tag{1a}$$

$$d_{(\pm)} = \frac{\log\left(\frac{x}{K}\right) + \left(r \pm \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, \tag{1b}$$

and where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\, dy$ is the standard normal distribution. The Black-Scholes price of the corresponding put option is given by $P(t, S(t), K, T)$, where

$$P(t, x, K, T) = \Phi(-d_{(-)})Ke^{-r\tau} - \Phi(-d_{(+)})x. \tag{2}$$

Moreover the put-call parity identity holds:

$$C(t, S(t), K, T) - P(t, S(t), K, T) = S(t) - Ke^{-r\tau}. \tag{3}$$

We have also seen that in the Black-Scholes model the number of shares of the stock in the hedging (REPLICATING) portfolio of standard European derivative is given by

$$h_S(t) = \Delta(t, S(t)), \quad \Delta(t,x) = \Phi\left(d_{(+)}(x)\right)$$

Two more definition to know for the exam:

Definition 6.9 (implied volatility)

Let $\widetilde{C}(t)$ be the market price at time $t < T$ of the European call with strike K and maturity T. If $\widetilde{C}(t) \in ((S(t)-Ke^{-r\tau})_+, S(t))$, the **implied volatility** σ_{imp} of the call option is the unique value of the volatility parameter σ such that

$$C(t, S(t), K, T, \sigma_{\text{imp}}) = \widetilde{C}(t).$$

Definition 6.10 (risk-neutral price of ZCB)

Let $\{r(t)\}_{t\in[0,S]}$ be a stochastic process modeling the spot interest rate of the ZCB market, where we assume that that $r(0) = r_0$ is a deterministic constant. Then

$$B(0,T) = \mathbb{E}[d(T)] = \mathbb{E}[\overbrace{e^{-\int_0^T r(s)\,ds}}^{\text{DISCOUNT PROCESS AT } t=0}], \quad 0 < T < S, \tag{4}$$

1

is called the risk-neutral value (at time $t = 0$) of the ZCB with maturity T and face value 1.

Exercises

Exercise 6.14

Compute the Black-Scholes price and the self-financing hedging portfolio of the physically-settled binary call. **Addendum:** Compute the probability that the derivative expire in the money and the expected return for the buyer. ANSWER: $\Pi_Y(t) = S(t)\Phi(d_{(+)})$, $h_S(t) = \Phi(d_{(+)}) + \frac{\phi(d_{(+)})}{\sigma\sqrt{\tau}}$.

Exercise 6.16 (solution in the book)

Consider the European derivative with pay-off $Y = S(T)(S(T)-K)$ and time of maturity T, where $K > 0$ is a constant. Compute the Black-Scholes price $\Pi_Y(t)$ of this derivative and the self-financing hedging portfolio. Finally, assume $S(0) = K$, $r > 0$ and compute the expected rate of return of a constant portfolio with 1 share of this derivative.

Exercise 6.41

A European derivative pays 1 if $S(T) > K$ and -1 otherwise, where $K > 0$. Determine K such that $\Pi_Y(0) = 0$. ANSWER: $K = S(0)e^{(r-\frac{\sigma^2}{2})T}$.

Exercise 6.42

Let $0 < L < K$. A European style derivative on a stock with maturity $T > 0$ pays nothing to its owner when $S(T) > K$, while for $S(T) < K$ it lets the owner choose between 1 share of the stock and the fixed amount L. (a) Draw the pay-off function of the derivative. (b) Compute the Black-Scholes price of the derivative. (c) Compute the number of shares of the stock in the hedging self-financing portfolio. ANSWER (c): $h_S(t) = \Delta(t, S(t))$, where $\Delta(t,x) = \Phi(d_2(K)) - \Phi(d_2(L)) - \phi(d_2(K))/\sigma\sqrt{\tau}$, where $d_2(a) = d_1(a) - \sigma\sqrt{\tau}$ and $d_1(a) = (\log\frac{a}{x} - (r - \frac{\sigma^2}{2})\tau)/\sigma\sqrt{\tau}$.

Solution to Exercise 4.6

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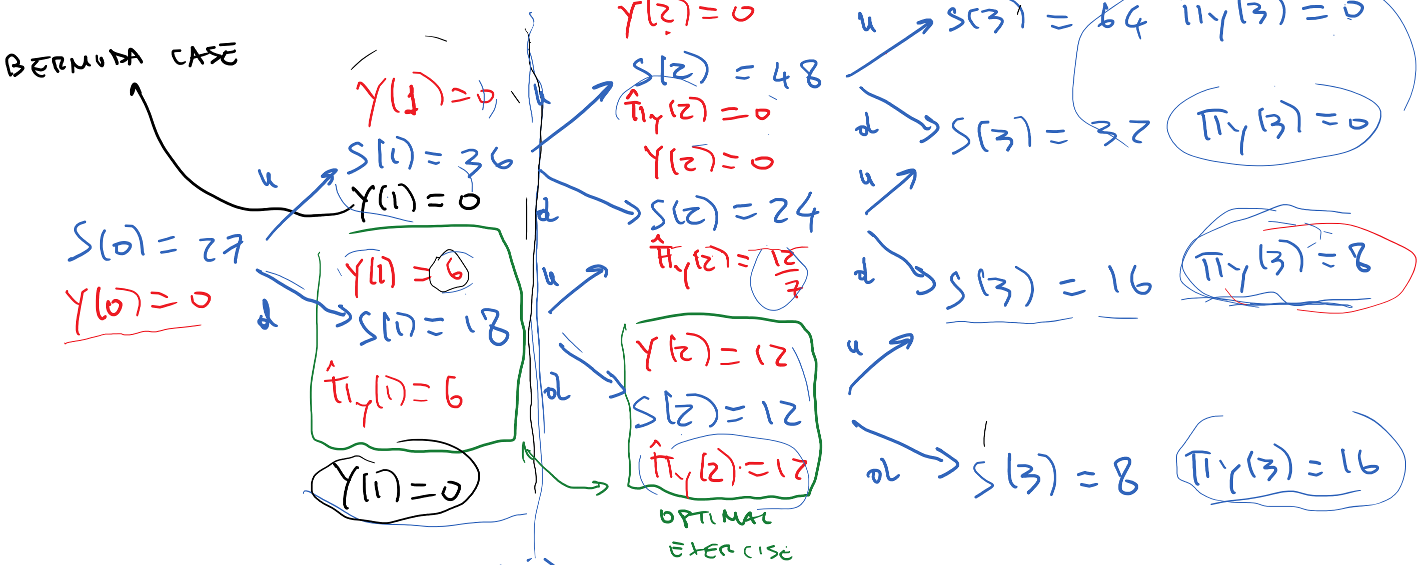
FIRST THING TO DO IS ALWAYS DRAW THE BINOMIAL TREE OF THE STOCK PRICE



EUROPEAN CASE (i)

$$\Pi_Y(3) = Y = (24 - S(3))_+$$

$$\rightarrow \Pi_Y(0) = e^{-3r} \sum_{x \in \{u,d\}^3} (q_u)^{N_u(x)} (q_d)^{N_d(x)} Y(x)$$

$$q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{7/6 - 2/3}{4/3 - \frac{2}{3}} = \frac{1/2}{2/3} = \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}$$

$$q_d = 1 - q_u = \frac{1}{4}$$

$$Y(u,d,d) = Y(d,u,d) = Y(d,d,u) = 8$$

$$Y(d,d,d) = 16$$

$$\Pi_Y(0) = \left(\frac{6}{7}\right)^3 \left[8\left(q_u (q_d)^2\right) \cdot 3 + 16 (q_d)^3 \right]$$

$$\Pi_Y(0) = \left(\frac{6}{7}\right)^3 \left[8 \left(3\, q_u (q_d)^2 \right) + 16 (q_d)^3 \right]$$

$$= \left(\frac{6}{7}\right)^3 \left[24 \cdot \frac{3}{4}\left(\frac{1}{4}\right)^2 + 16 \left(\frac{1}{4}\right)^3 \right] = \frac{299}{343}$$

AMERICAN CASE (ii)

$$Y(t) = \left(24 - S(t)\right)_+$$

$$\hat{\Pi}_Y(t) = \max\left[Y(t),\ e^{-r}\left(q_u \hat{\Pi}_Y^u(t+1) + q_d \hat{\Pi}_Y^d(t+1)\right)\right]$$

$$\hat{\Pi}_Y(2,u,d) = \hat{\Pi}_Y(2,d,u) = \max\left[0,\ \left(\frac{6}{7}\right)\left(q_u \cdot 0 + q_d^{1/4}\, 8\right)\right]$$

$$= \frac{12}{7}$$

$$\hat{\Pi}_Y(2,d,d) = \max\left[12,\ \frac{6}{7}\left(\frac{3}{4}\, 8 + \frac{1}{4}\cdot 16\right)\right]$$

$$= \max\left[12,\ \frac{6}{7}\left(6 + 4\right)\right] = 12$$

$$\hat{\Pi}_Y(1,u) = \max\left(0,\ \frac{6}{7}\ \frac{1}{4}\cdot\frac{\overset{3}{\cancel{12}}}{7}\right) = \frac{18}{49} \leftarrow$$

0 IN THE BERMUDA CASE

$$\hat{\Pi}_Y(1,d) = \max\left[(6),\ \frac{6}{7}\left(\frac{3}{4}\,\frac{12}{7} + \frac{1}{4}\,12\right)\right]$$

$$= \max\left(6,\ \frac{6}{7}\left(\frac{9}{7} + 3\right)\right)$$

$$= \max\left(6,\ \frac{6}{7}\cdot\frac{30}{7}\right)$$

$$= \max\left(6,\ \frac{180}{49}\right) = 6$$

PRICE IN THE BERMUDA CASE

$$\hat{\Pi}_Y(0) = \max\left(0,\ \frac{6}{7}\left[\frac{3}{4}\,\frac{18}{49} + \frac{1}{4}\cdot 6\right]\right) = 522$$

$$\hat{\Pi}_Y(0) = \max\left(0, \underbrace{\frac{6}{7}\left(\frac{3}{4}\frac{\tilde{16}}{49} + \frac{1}{4}\cdot 6\right)}\right) = \frac{522}{343}$$

$$> \Pi_Y(0)$$

BERMUDA CASE (iii)

COMPUTE AS IN THE AMERICAN CASE

BUT WITH $Y(1,u) = Y(1,d) = 0$

$$\tilde{\Pi}_Y(0) = \frac{351}{343} \qquad \Pi_Y(0) < \tilde{\Pi}_Y(0) < \hat{\Pi}_Y(0)$$

CASH FLOW AT THE OPTIMAL EXERCISE TIME AT $t=2$ FOR THE AMERICAN PUT:

$$C(2) = \hat{\Pi}_Y(2) - e^{-r}\left[q_u \hat{\Pi}^u_Y(3) + q_d \hat{\Pi}^d_Y(3)\right]$$

$$= 12 - \frac{6}{7}\left[\frac{3}{4}8 + \frac{1}{4}\cdot 16\right]$$

$$= 12 - \frac{6}{7}\left[6+4\right] = 12 - \frac{60}{7} = \frac{24}{7}$$

EXPECTED RETURN IN THE EUROPEAN CASE

$$\mathbb{E}[R] = \mathbb{E}\left[\Pi_Y(3) - \Pi_Y(0)\right]$$

$$= \mathbb{E}\left[\Pi_Y(3)\right] - \Pi_Y(0)$$

$$= \frac{1}{8}\cdot 16 + \frac{3}{8}\cdot 8 - \frac{299}{343} = 5 - \frac{299}{343} > 0$$

BREAK UNTIL 3.15 PM

Solutions to Exercise 6.14

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$$Y = S(T)\, H(\underbrace{S(T) - K})$$

$$g(z) = \underbrace{z H(z-K)} \qquad Y = g(S(T))$$

$$\Pi_Y(t) = v(t, S(t)) \quad \text{WHERE}$$

$$v(t,x) = e^{-r\tau} \int_{\mathbb{R}} g(\underbrace{x e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau} y}}_{z}) e^{-\frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}}$$

$$\tau = T - t$$

$$= e^{-r\tau} \int_{\mathbb{R}} x e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau} y}\, H(\underbrace{x e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}y} - K}) e^{-\frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}}$$

$$x e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}y} - K > 0$$

$$y > \frac{\log\frac{K}{x} - (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} = -d_{(-)}$$

$$v(t,x) = x e^{-r\tau} e^{(r-\frac{\sigma^2}{2})\tau} \int_{-d_{(-)}}^{\infty} e^{\sigma\sqrt{\tau}y} \cdot e^{-\frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}}$$

$$= x \cancel{e^{-\frac{\sigma^2}{2}\tau}} \int_{-d_{(-)}}^{\infty} e^{\sigma\sqrt{\tau} - \frac{1}{2}y^2 - \frac{1}{2}\sigma^2\tau}\, \cancel{e^{\frac{1}{2}\sigma^2\tau}} \frac{dy}{\sqrt{2\pi}}$$

$$= x \cancel{e^{-\frac{\sigma^2}{2}\tau}} \int_{-d_{(-)}}^{\infty} e^{\sigma\sqrt{\tau} y - \frac{1}{2}y^2 - \frac{1}{2}\sigma^2\tau} \cancel{e^{\frac{1}{2}\sigma^2\tau}} \frac{dy}{\sqrt{2\pi}}$$

$$= x \int_{-d_{(-)}}^{\infty} e^{-\frac{1}{2}\underbrace{(y - \sigma\sqrt{\tau})^2}_{z}} \frac{dy}{\sqrt{2\pi}}$$

$$= x \int_{-d_{(-)} - \sigma\sqrt{\tau}}^{\infty} e^{-\frac{1}{2}z^2} \frac{dz}{\sqrt{2\pi}} = x \int_{-\infty}^{\overbrace{d_{(-)} + \sigma\sqrt{\tau}}^{d_{(+)}}} e^{-\frac{1}{2}z^2} \frac{dz}{\sqrt{2\pi}}$$

$$= x \, \Phi(d_{(+)}) \leftarrow$$

$$\Pi_V(t) = v(t, S(t)) = S(t)\, \Phi(d_{(+)})$$

REPLICATING PORTFOLIO :

$$h_S(t) = \Delta(t, S(t)) \quad , \quad \Delta(t,x) = \partial_x v(t,x)$$

$$\Delta(t,x) = \partial_x \left[x \, \Phi(d_{(+)}) \right]$$

$$= \Phi(d_{(+)}) + x \, \partial_x \left[\Phi(d_{(+)}) \right]$$

$$\left\{ d_{(+)}(x) = \left[\frac{\log \frac{x}{K} - (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \right] = \frac{\log x}{\sigma\sqrt{\tau}} + \text{CONSTANT} \right\}$$

$$= \Phi(d_{(+)}) + x \, \Phi'(d_{(+)}(x)) \, \partial_x d_{(+)}(x)$$

$$= \Phi(d_{(+)}) + \cancel{x} \, \varphi(d_{(+)}(x)) \frac{1}{\cancel{x}\, \sigma\sqrt{\tau}}$$

ADDENDUM:

$$\mathbb{P}(Y > 0) = \mathbb{P}(S(T) > K)$$

$\mathbb{P}$ IS THE PHYSICAL PROBABILITY, HENCE WE HAVE TO USE THE EXPRESSION OF $S(t)$ IN $\mathbb{P}$:

$$S(t) = \begin{cases} S(0)e^{\alpha t + \sigma W(t)} & \text{in } \mathbb{P} \\ S(0)\, e^{\left(r - \frac{\sigma^2}{2}\right)t + \sigma W^{(q)}(t)} & \text{in } \mathbb{P}_q \end{cases}$$

$$\mathbb{P}(S(T) > K) = \mathbb{P}\Big(S(0)\, e^{\alpha T + \sigma \underbrace{W(T)}_{\sim N(0,T)}} > K\Big)$$

$$= \mathbb{P}\left(S(0)\, e^{\alpha T + \sigma\sqrt{T}\frac{W(T)}{\sqrt{T}}} > K\right)$$

$$\frac{W(T)}{\sqrt{T}} = G \in N(0,1)$$

$$S(0)\, e^{\alpha T + \sigma\sqrt{T}\, G} > K \iff G > \frac{\log\frac{K}{S(0)} - \alpha T}{\sigma\sqrt{T}} = a$$

$$\mathbb{P}(Y > 0) = \mathbb{P}(G > a)$$

$$= \int_a^{\infty} e^{-\frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}} = \Phi(-a)$$

$$\mathbb{E}[R] = \mathbb{E}\left[\Pi_Y(T) - \Pi_Y(0)\right] = \mathbb{E}[Y] - \Pi_Y(0)$$

$$\mathbb{E}[Y] = \mathbb{E}\left[S(T)H(S(T)-K)\right]$$

$$= \int_{\mathbb{R}} x\, H(x-K)\, f_{S(T)}(x)\, dx$$

$$= \int_K^{\infty} x\, \underbrace{f_{S(T)}}_{\text{EQ. (6.14)}}(x)\, dx = \ldots$$

COMPLETE THE CALCULATION

Solution to Exercise 6.42

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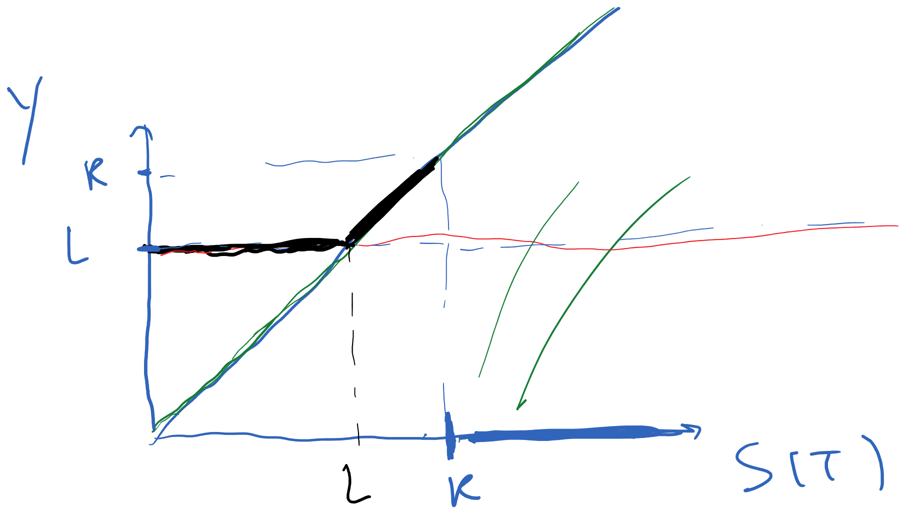
Solution



$$Y = g(S(T)) = \begin{cases} L & \text{IF } 0 < S(T) < L \\ S(T) & \text{IF } L < S(T) \leq K \\ 0 & \text{IF } S(T) > K \end{cases}$$

$$\Pi_Y(t) = v(t, S(t))$$

$$v(t,x) = e^{-r\tau} \int_{\mathbb{R}} g\Big(\underbrace{x\, e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}\, y}}_{z}\Big)\, e^{-\frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}}$$

$$z < L \iff x e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}\, y} < L$$

$$y < \frac{\log \frac{L}{x} - (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} = -d_1(L)$$

$$z < K \iff y < -d_1(K) \qquad z > K \iff y > -d_1(K)$$

$$z > L \iff y > \frac{\log \frac{L}{x} - (r - \frac{\sigma^2}{2})\tau}{\ } = -d_1(L)$$

$$x > L \iff Y > \frac{\log\frac{L}{x} - \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} = -d_1(L)$$

THEN

$$v(t,x) = e^{-r\tau} \int_{-\infty}^{-d_1(L)} L\, e^{-\frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}} = \Phi\left(-d_1(L)\right)$$

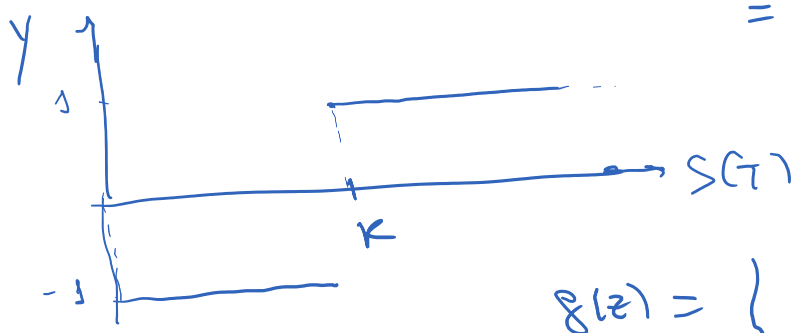
$$+ e^{-r\tau} \int_{-d_1(L)}^{-d_1(K)} x\, e^{\left(r - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\, y}\, e^{-\frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}}$$

$= \text{----}$

EXERCISE 6.41

$$Y = \begin{cases} 1 & \text{IF } S(T) > K \\ -1 & \text{IF } 0 < S(T) \le K \end{cases}$$

$$= g(S(T))$$



$$g(z) = \begin{cases} 1 & \text{IF } z > k \\ -1 & \text{IF } z < k \end{cases}$$

$$\Pi_Y(t) = v(t, S(t))$$

$$v(t,x) = e^{-r\tau} \int_{\mathbb{R}} g\Big(\underbrace{x e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}\, y}}_{z}\Big) e^{-\frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}}$$

$$z < K \iff y < -d_1(k)$$

$$z > k \iff y > -d_1(k)$$

$$v(t,x) = e^{-r\tau}\left[\int_{-\infty}^{-d_1(k)} (-1)\, e^{-\frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}} + \int_{-d_1(k)}^{\infty} 1\, e^{-\frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}}\right]$$

$$= e^{-r\tau}\left[-\Phi(-d_1(k)) + \int_{-\infty}^{d_1(k)} e^{-\frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}}\right]$$

$$= e^{-r\tau}\left[-\Phi(-d_1(k)) + \Phi(d_1(k))\right]$$

$$= e^{-r\tau}\left[-\Phi(-d_1(k)) + \Phi(d_1(k)) \right]$$

$$\left\{ \Phi(x) + \Phi(-x) = 1 \right\}$$

$$= e^{-r\tau}\left[2\Phi(d_1(k)) - 1 \right]$$