Exam for the course "Options and Mathematics" (CTH[*MVE095*], GU[*MMG810*]) 2020/21

For questions call the examiner at +46 (0)31 772 35 62

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REMARKS: (1) All aids permitted, however you must work alone (2) Give all details and explain all steps in your solutions (3) Write as clear as possible: if some step is not clearly readable it will assumed to be wrong! (4) See the course homepage for instructions on how to submit the exam.

Part I

1. Prove that any European style derivative in an arbitrage-free binomial market can be hedged by a self-financing predictable portfolio invested in the underlying stock and the risk-free asset (max 2 points) **Solution.** See Theorem 3.3

2. Give and explain the definition of implied volatility of European call options (max 2 points) **Solution.** See Definition 6.9

3. Assume that the market is arbitrage-free. Let $P(t, S(t), K, T)$ be the price at time $t \in [0, T]$ of the European put with strike K and maturity T and $\widehat{P}(t, S(t), K, T)$ be the price of the corresponding American put. The underlying stock pays no dividend in the interval $[0, T]$. Decide whether the following statements are true or false and explain your answer (max 2 points):

   (a) If $\widehat{P}(0, S(0), K, T) = P(0, S(0), K, T)$, then $\widehat{P}(t, S(t), K, T) = P(t, S(t), K, T)$ for all $t \in [0, T]$.

   (b) If the risk-free rate r is negative, then $\widehat{P}(t, S(t), K, T) = P(t, S(t), K, T)$, for all $t \in [0, T]$.

   Solution. (a) is true. In fact, if the European and the American put has the same value at time $t = 0$, it means that the option of early exercise in the future is worthless. Thus the two derivatives have the same value at all times. Alternative argument: at time $t = 0$ open a portfolio with 1 share of the American put and -1 share of the European put. The initial value of this portfolio is zero and the investor can keep it open until time T (since the investor has the long position on the American put and thus can decide not to exercise it until maturity). If at some time $t \in [0, T)$ the American put becomes more valuable than the European one, then at this time the investor can take a short position on the American put and a long position on the European put, thereby closing the portfolio, and cashing the difference $\widehat{P}(t, S(t), K, T) - P(t, S(t), K, T)$. This strategy is an arbitrage. (b) is also true. The simplest argument is by using the put call parity:

   $$\widehat{P}(t, S(t), K, T) \geq P(t, S(t), K, T) = C(t, S(t), K, T) - S(t) + Ke^{-r(T-t)} > K - S(t), \quad t \in [0, T),$$

where we used that $e^{-r(T-t)} > 1$ for $r > 0$ and $t \in [0, T)$. Hence if the American put is in the money, that is $S(t) < K$, we have $\widehat{P}(t, S(t), K, T) > (K - S(t))_+$. Therefore there is no optimal exercise time before maturity and so the American pus is equivalent to the European put.

Part II

1. Let $S(t) > 0$ be the price at time t of a non-dividend paying stock. A European style derivative with expiration $T = 3$ months has the pay-off depicted in the figure in the next page (blue line). Find a constant portfolio on *European put options* that replicates the value of the derivative (max 2 points). Assuming that the stock price is given by a geometric Brownian motion with zero mean of log-return, 50 % volatility and $S(0) = 1$, compute the probability that the derivative expires in the money. Express the result in terms of the standard normal distribution (max 2 points).

   Solution. The sought portfolio is $(P(1), -P(2), -P(3), P(4), P(5), -2P(6), P(7))$, where $P(K)$ is the European put with strike K. The probability that it expires in the money is

   $$\mathbb{P}(Y > 0) = \mathbb{P}(1 < S(T) < 4) + \mathbb{P}(5 < S(T) < 7).$$

   In the given Black-Scholes market ($\alpha = 0, \sigma = 1/2, T = 1/4, S(0) = 1$) we have

   $$S(T) = S(0)e^{\alpha T + \sigma W(T)} = S(0)e^{\alpha T + \sigma\sqrt{T}G} = e^{\frac{1}{4}G},$$

   where $G = W(T)/\sqrt{T} \in N(0, 1)$. Thus

   $$\begin{aligned}\mathbb{P}(Y > 0) &= \mathbb{P}(1 < e^{G/4} < 4) + \mathbb{P}(5 < e^{G/4} < 7) = \mathbb{P}(0 < G < 4\log 4) + \mathbb{P}(4\log 5 < G < 4\log 7)\\ &= \Phi(4\log 4) - 1/2 + \Phi(4\log 7) - \Phi(4\log 5).\end{aligned}$$

2. Consider a 3 period binomial model with the following parameters:

   $$S(0) = 8, \quad u = \log 2, \quad d = -\log 2, \quad r = 0, \quad p \in (0, 1).$$

   Consider also a European style derivative $\mathcal{U}$ in this market such that $\mathcal{U}$ expires worthless if the stock price exceeds $Q = 30$ or falls below $L = 3/2$ at some time $t \in \{1, 2, 3\}$ and which otherwise pays the amount

   $$Y = \max\{S(t), t = 0, 1, 2, 3\} - \min\{S(t), t = 0, 1, 2, 3\}.$$

   Compute the binomial price of the derivative at time $t = 0$ (max 2 points). Find the value of p which maximizes the probability that $\mathcal{U}$ expires in the money and the value of p which maximizes the probability that the return on 1 share of $\mathcal{U}$ be positive (max 2 points).

Solution. The binomial tree of the stock price is

$$
\begin{tikzcd}
 & & & \boxed{S(3)=64} \\
 & & \boxed{S(2)=32} \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & S(1)=16 \arrow[ur, "u"] \arrow[dr, "d"] & & S(3)=16 \\
S(0)=8 \arrow[ur, "u"] \arrow[dr, "d"] & & S(2)=8 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & S(1)=4 \arrow[ur, "u"] \arrow[dr, "d"] & & S(3)=4 \\
 & & S(2)=2 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & & \boxed{S(3)=1}
\end{tikzcd}
$$

The prices within a box are above/below the upper/below barrier. Hence the pay-off is zero along the paths that hit these prices. It follows that

$$
\begin{aligned}
&Y(u,d,u) = 16-8 = 8, \quad Y(u,d,d) = 16-4 = 12, \quad Y(d,u,u) = 16-4 = 12, \\
&Y(d,u,d) = 8-4 = 4, \quad Y(d,d,u) = 8-2 = 6
\end{aligned}
$$

while $Y = 0$ along the other paths. To compute the price of the derivative at time $t = 0$ we use the formula

$$\Pi_Y(0) = e^{-rN} \sum_{x\in\{u,d\}^N} (q_u)^{N_u(x)}(q_d)^{N_d(x)} Y(x).$$

In this case we have $N = 3$, $r = 0$, $q_u = 1/3$, $q_d = 2/3$, hence

$$
\begin{aligned}
\Pi_Y(0) &= (1/3)^2(2/3)Y(u,d,u) + (1/3)(2/3)^2 Y(u,d,d) + (1/3)^2(2/3)Y(d,u,u) \\
&\quad + (1/3)(2/3)^2 Y(d,u,d) + (1/3)(2/3)^2 Y(d,d,u) = \frac{128}{27}
\end{aligned}
$$

The probability that the derivative expires in the money is

$$
\begin{aligned}
\mathbb{P}(Y>0) =& \mathbb{P}(u,d,u) + \mathbb{P}(u,d,d) + \mathbb{P}(d,u,u) + \mathbb{P}(d,u,d) + \mathbb{P}(d,d,u) \\
&= 2p^2(1-p) + 3p(1-p)^2 := F(p).
\end{aligned}
$$

The function F has a maximum at $p \in (0,1)$ such that $F'(p) = 0$, that is $3p^2 - 8p + 3 = 0$, which gives

$$p = \frac{4-\sqrt{7}}{3}.$$

The return on the derivative is positive only along the paths where $Y > \Pi_Y(0)$, that is $(u,d,u), (u,d,d), (d,u,u), (d,d,u)$. Hence

$$\mathbb{P}(R>0) = \mathbb{P}(u,d,u) + \mathbb{P}(u,d,d) + \mathbb{P}(d,u,u) + \mathbb{P}(d,d,u) = 2p^2(1-p) + 2p(1-p)^2 = 2p(1-p)$$

which has a maximum at $p = 1/2$.

3. Show that in a Black-Scholes market with non-negative risk-free rate and no dividends, the Asian call with strike K and maturity $T > 0$ is, at time $t = 0$, less valuable than the standard European call on the same stock and with the same strike and maturity (max 4 points).

HINT: You need the Jensen inequality for integrals:

$$\phi\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right) \leq \frac{1}{b-a}\int_a^b \phi(f(x))\,dx,$$

for all $b > a$, $f : \mathbb{R} \to \mathbb{R}$ and convex functions $\phi : \mathbb{R} \to \mathbb{R}$.

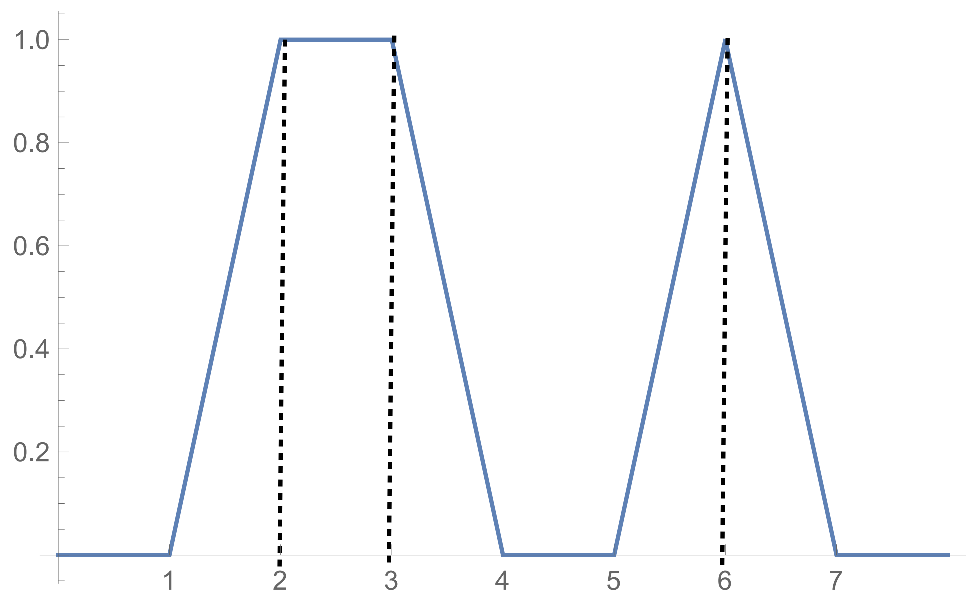
Solution. By the convexity of the function $\phi(x) = (x)_+$ and the Jensen inequality in the hint, we have

$$\Pi_{\mathrm{AC}}(0) = e^{-rT}\mathbb{E}_q\left[\left(\frac{1}{T}\int_0^T (S(t)-K)\,dt\right)_+\right] \leq \frac{e^{-rT}}{T}\mathbb{E}_q\left[\int_0^T (S(t)-K)_+\,dt\right]$$
$$= \frac{e^{-rT}}{T}\int_0^T \mathbb{E}_q[(S(t)-K)_+]\,dt = \frac{e^{-rT}}{T}\int_0^T e^{rt}e^{-rt}\mathbb{E}_q[(S(t)-K)_+]\,dt.$$

Since for $r \geq 0$ the (Black-Scholes) value of the call option is decreasing with maturity, then $e^{-rt}\mathbb{E}_q[(S(t)-K)_+] = C(0,S_0,K,t) < C(0,S_0,K,T)$, for $t \in [0,T)$. Thus

$$\Pi_{\mathrm{AC}}(0) < \frac{e^{-rT}}{T}C(0,S_0,K,T)\int_0^T e^{rt}\,dt = \frac{1-e^{-rT}}{rT}C(0,S_0,K,T).$$

As the function $x \to (1-e^{-x})/x$ is bounded by 1 for $x \geq 0$, we obtain in particular $\Pi_{\mathrm{AC}}(0) < C(0,S_0,K,T)$.



Remark: For $S(T) > 7$ the pay-off is identically zero.