Exam for the course "Options and Mathematics" (CTH[*MVE095*], GU[*MMG810*]) 2019/20

April 8^{th}, 2020

REMARKS: (1) All aids permitted, however you must work alone (2) Minor errors in the calculations will be forgiven, but remember that fractions look nicer when you simplify them!

Part I: The number of points assigned to the tasks in this part will be based on the amount of details provided in the solution. Use your own words and explain each step in the proof, in more details than in the lecture notes. No point will be awarded by writing "just" what is written in the lecture notes.

1. Assume that the market is frictionless, arbitrage free and that the assets pay no dividend. Prove that the price of the call option is a non-increasing convex function of the strike (max 2 points).

2. Give a complete proof of the formula for the Black-Scholes price at time $t = 0$ of standard European derivatives on a dividend paying stock (max 2 points).

3. Give and explain the definition of binomial price of European derivatives (max 2 points).

4. A forward contract with delivery price K and maturity T on an asset $\mathcal{U}$ is a European style derivative stipulated by two parties which compels one party to sell, and the other party to buy, the asset $\mathcal{U}$ at time T for the price K. Assume that the forward contact is stipulated at time $t = 0$. Decide whether the following statements are true or false and explain your answer (max 2 points):

   (a) The fair value of the forward contact is zero;

   (b) The fair value of the delivery price K that the two parties should write on the contract is $\Pi^{\mathcal{U}}(0)$, where $\Pi^{\mathcal{U}}(t)$ is the price of the underlying asset $\mathcal{U}$ at time t.

   Solution. (a) is true: both parties have the same right/obligation, hence none of them has to pay a premium to the other. (b) is false: the fair value of the delivery price K is $\Pi^{\mathcal{U}}(0)e^{rT}$, assuming say that the risk-free rate is constant. This is because the seller of the asset could invest the quantity $\Pi^{\mathcal{U}}(0)$ in the money market at time $t = 0$ if the asset was sold on the spot, hence the fair value of the K is the value of this investment at time T.

Part II: Do not skip calculations in the exercises and write as clear as possible. If some portion of the solution is not clearly readable, it will be assumed to be wrong.

1. Find a constant portfolio consisting of European calls and/or puts that replicates the European derivative with maturity T and pay-off Y depicted in the next page (max 4 points).

   Solution: $(-2P(1), 2P(3), 3/2C(4), -3C(6), 3/2C(8))$, where $C(K), P(K)$ are the call and put with strike K and maturity T. Remark: other solutions are possible.

2. Consider a 2-period binomial model with $r > 0$, $u > 0$, $d = -u$ and risk-neutral probability vector $q_u = e^{-r}$, $q_d = 1 - e^{-r}$. Let $S(t)$ be the binomial price of the stock. Show that when the price of the stock goes down in the first step the earlier exercise at time $t = 1$ of the American put with strike $K = S(0)$ and maturity $T = 2$ is always optimal, for all values of $r > 0$ (max 3 points). Find the price of the American put at time $t = 0$ (max 1 point).

   Solution. The binomial tree of the stock price is

$$
\begin{tikzcd}
 & & S(0)e^{2u} \Rightarrow \widehat{\Pi}_{\text{put}}(2) = 0 \\
 & S(0)e^{u} \arrow[ur, "u"] \arrow[dr, "d"] & \\
S(0) \arrow[ur, "u"] \arrow[dr, "d"'] & & S(2) = S(0) \Rightarrow \widehat{\Pi}_{\text{put}}(2) = 0 \\
 & S(0)e^{-u} \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & S(0)e^{-2u} \Rightarrow \widehat{\Pi}_{\text{put}}(2) = S(0)(1 - e^{-2u})
\end{tikzcd}
$$

   where we used that $\widehat{\Pi}_{\text{put}}(2) = (S(0) - S(2))_+$. From this it is clear that $\widehat{\Pi}_{\text{put}}(1, u) = 0$. If $S(1) = S(0)e^{-u}$, the price of the put is

$$\widehat{\Pi}_{\text{put}}(1, d) = \max(S(0)(1-e^{-u}), e^{-r} q_d S(0)(1-e^{-2u})) = S(0)\max((1-e^{-u}), e^{-r}(1-e^{-r})(1-e^{-2u}))$$

   Thus to show that it is optimal to exercise at this point we have to show that $1 - e^{-u} > e^{-r}(1 - e^{-r})(1 - e^{-2u})$ for all $r > 0$, that is

$$(1 + e^{-u})(1 - e^{-r})e^{-r} < 1, \quad \text{for all } r > 0.$$

   Since $q_u, q_d \in (0, 1)$, then the market is arbitrage free and therefore $u > r$. It follows that the left hand side of the inequality is bounded by $(1 + e^{-r})(1 - e^{-r})e^{-r} = (1 - e^{-2r})e^{-r} < 1$, which proves the claim (3 points). In particular $\widehat{\Pi}_{\text{put}}(1, d) = S(0)(1 - e^{-u})$, hence the price of the American put at time $t = 0$ is

$$\widehat{\Pi}_{\text{put}}(0) = e^{-r} q_d S(0)(1 - e^{-u}) = S(0)e^{-r}(1 - e^{-r})(1 - e^{-u}).$$

   This concludes the second part of the exercise (1 point).

3. Compute the Black-Scholes price at time $t < T$ of the European derivative with pay-off $Y = S(T)^{\delta} H(S(T) - K)$ and maturity T, where $S(t)$ is the price of the underlying stock,

$\delta > 0$ is a constant and $H(z)$ is the Heaviside function (max 2 points). Derive a parity relation at $t = 0$ satisfied by this derivative and the one with pay-off $Z = S(T)^\delta H(K - S(T))$ (max 2 points).

Solution. The pay-off function is $g(z) = z^\delta H(z - K)$. The Black-Scholes price is $\Pi_Y(t) = v(t, S(t))$, where

$$\begin{aligned} v(t,x) &= e^{-r\tau} \int_{\mathbb{R}} g\left(xe^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}y}\right) e^{-\frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}} \\ &= e^{-r\tau} \int_{-d_2}^{\infty} x^\delta e^{\delta(r-\frac{\sigma^2}{2})\tau+\delta\sigma\sqrt{\tau}y-\frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}} \end{aligned}$$

where

$$d_2 = \frac{\log\frac{x}{K} + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}, \quad \tau = T - t.$$

Proceeding by computing the integral as usual we find

$$v(t,x) = x^\delta e^{-(1-\delta)(r+\frac{1}{2}\sigma^2\delta)\tau}\Phi(d_1), \quad d_1 = d_2 + \delta\sigma\sqrt{\tau}.$$

This completes the first part of the exercise (2 points). As to the parity relation, we notice that $H(z - K) + H(K - z) = 1$, hence

$$\Pi_Y(0) + \Pi_Z(0) = e^{-rT}\widetilde{\mathbb{E}}[Y + Z] = e^{-rT}\widetilde{\mathbb{E}}[S(T)^\delta].$$

Using that

$$S(T) = S(0)e^{(r-\frac{\sigma^2}{2})T+\sigma\widetilde{W}(T)}$$

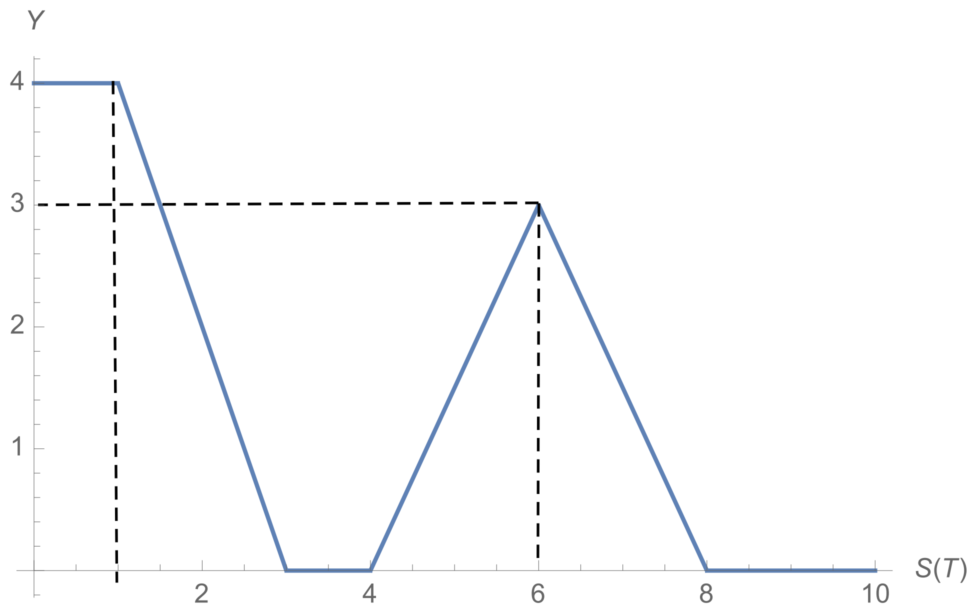
in the risk-neutral probability, we find

$$\Pi_Y(0) + \Pi_Z(0) = S(0)^\delta e^{-rT} e^{(r-\frac{\sigma^2}{2})\delta T}\widetilde{\mathbb{E}}[e^{\sigma\delta\sqrt{T}G}]$$

where G is a standard normal random variable in the risk-neutral probability. Hence

$$\Pi_Y(0) + \Pi_Z(0) = S(0)^\delta e^{-rT} e^{(r-\frac{\sigma^2}{2})\delta T} \int_{\mathbb{R}} e^{\sigma\delta\sqrt{T}x-\frac{1}{2}x^2} \frac{dx}{\sqrt{2\pi}} = S(0)^\delta e^{-(r+\frac{\sigma^2}{2}\delta)T(1-\delta)}.$$

This concludes the second part of the exercise (2 points).



Remark: For $S(T) > 10$ the pay-off is identically zero.