Foundations of Probability Theory
(MVE140 – MSA150)

Saturday 15th of January 2022 examination questions

You are allowed to use a dictionary (to and from English) and up to a maximum of one double-sided page of your own hand-written notes. This examination has five problems with a maximum of 20 credit points for a fully satisfactory solution, so the maximal total is 100 credit points. To pass the course, you need to score at least 40 points (with any bonus you have) (50 if you are a PhD student).

Examination Questions

1. Let $\mathcal{A}$ is the system of semi-open segments $\{[0, n),\ n \in \mathbb{Z}_+\}$ in $\mathbb{R}$.

   i) Is $\mathcal{A}$ a σ-field?

   ii) Describe the σ-field $\mathcal{F}$ generated by $\mathcal{A}$.

   iii) Which functions from $\mathbb{R}$ to $\mathbb{R}$ are $\mathcal{F}$-measurable?

   iv) Establish that the set of measurable functions has the power of $\mathbb{R}^{\mathbb{N}}$ (which is continuum).

   Solution.

   i) No. For instance, $[0, 1)^{\mathsf{c}} = (-\infty, 0) \cup [1, \infty) \notin \mathcal{A}$.

   ii) Together with $\emptyset$ and $\mathbb{R}$, $\mathcal{F}$ contains $(-\infty, 0)$ and intervals $[n, n+1)$ for $n \in \mathbb{Z}_+$, their unions and complements, in particular, $[m, n)$, $0 \le m < n$, and unbounded sets of the form $[n, \infty)$.

   iii) Only functions of the type $f(x) = f_0\, \mathbb{I}_{(-\infty,0)}(x) + \sum_{n=1}^{\infty} f_n\, \mathbb{I}_{[n,n+1)}(x)$ for some constants $f_0, f_1, \ldots$. In particular, they are right continuous. The bijection $f \leftrightarrow (f_0, f_1, \ldots)$ establish equivalence of these functions to $\mathbb{R}^{\mathbb{N}}$.

2. Let $X_1, X_2, X_3 \ldots$ be independent Bernoulli trials with success probability p and $S_k = X_1 + \ldots + X_k$. Let $m < n$.

i) Find the conditional probability mass function of $(X_1, \dots, X_n)$ given that $S_n = k$. That is, find

$$\mathbf{P}\{X_1 = a_1, \dots, X_n = a_n \bigm| S_n = k\}$$

for all vectors $(a_1, \dots, a_n)$ of zeros and ones. Identify the distribution by name and given an intuitive explanation of the answer.

ii) Find the conditional probability mass function $f_{S_m|S_n}(l,k)$ of S_m given $S_n = k$.

Solution.

i) When $S_n = k$, exactly k of X_i's $1 \le i \le n$ are success (1's). Therefore,

$$\begin{aligned}\mathbf{P}\{X_1 = a_1, \dots, X_n = a_n, S_n = k\} \\ = \mathbf{P}\{X_1 = a_1, \dots, X_n = a_n\} = p^k(1-p)^k \quad (1)\end{aligned}$$

for all $(a_1, \dots, a_n)$ such that $\sum_{i=1}^n a_i = k$ and 0 otherwise. Since sum S_n is Binomially $\mathsf{Bin}(n,p)$ distributed,

$$\mathbf{P}\{X_1 = a_1, \dots, X_n = a_n \bigm| S_n = k\} = \frac{p^k(1-p)^k}{\binom{n}{k}p^k(1-p)^k} = \frac{1}{\binom{n}{k}}$$

for all $(a_1, \dots, a_n)$ such that $\sum_{i=1}^n a_i = k$ and 0 otherwise. Since there are exactly $\binom{n}{k}$ of these, the distribution is uniform over such $(a_1, \dots, a_n) : \sum_{i=1}^n a_i = k$. The result is expected because of the symmetry of the probability (1) under permutations of a_i's.

ii) When $S_m = l$ and $S_n = k$, the sum $S' = X_{m+1} + \ldots + X_n$ is $k - l$. We have that $S' \sim \mathsf{Bin}(n-m, p)$ and it is independent of S_m, implying

$$\begin{aligned}f_{S_m|S_n}(l,k) &= \frac{\mathbf{P}\{S_m = l\}\mathbf{P}\{S' = k - l\}}{\mathbf{P}\{S_n = k\}} \\ &= \frac{\binom{m}{l}p^l(1-p)^{m-l}\binom{n-m}{k-l}p^{k-l}(1-p)^{n-m-k+l}}{\binom{n}{k}p^k(1-p)^{n-k}} = \frac{\binom{m}{l}\binom{n-m}{k-l}}{\binom{n}{k}}, \ l = 0, \dots, k.\end{aligned}$$

3. Variable η has density $f_\eta(y) = 3y^2\,\mathbb{I}_{0<y<1}$. For $y \in (0,1)$, given $\eta = y$, variable ξ has conditional density $f_{\xi|\eta}(x|y) = 2x/y^2\,\mathbb{I}_{0<x<y}$.

   i) Find the joint density $f_{\xi,\eta}(x,y)$ of (ξ,η). Be precise about where it is non-zero and check that it integrates to 1.

   ii) Find the conditional density $f_{\eta|\xi}(y|x)$, identify the conditional distribution by name.

   iii) Find $\mathbf{E}[\eta \mid \xi]$ and verify that $\mathbf{E}\,\mathbf{E}[\eta \mid \xi] = \mathbf{E}\,\eta$.

Solution. *i)* The joint density is positive in the triangle S bounded by $0 < y < 1$ and $0 < x < y$ having vertices $(0,0)$, $(0,1)$ and $(1,1)$:

$$f_{\xi,\eta}(x,y) = f_{\xi|\eta}(x|y) f_\eta(y) = 6x\,\mathbb{I}_S(x,y)$$

ii) The marginal density of ξ is

$$f_\xi(x) = \int 6x\,\mathbb{I}_S(x,y)\,dy = 6x(1-x),\ x \in (0,1).$$

Hence

$$f_{\eta|\xi}(y|x) = \frac{6x\,\mathbb{I}_S(x,y)}{6x(1-x)} = \frac{\mathbb{I}_{x<y<1}}{1-x},\ x \in (0,1),$$

i.e. η given $\xi = x$ is uniformly distributed on $[x,1]$. *iii)* The conditional expectation is the middle point for the uniform distribution on $[\xi,1]$: $\mathbf{E}[\eta \mid \xi] = (1+\xi)/2$, its expectation is

$$\mathbf{E}\,\mathbf{E}[\eta \mid \xi] = \int_0^2 (1+x)/2\,6x(1-x)\,dx = 3\int_0^1 (x-x^3)\,dx = 3/4 = \int_0^1 y\cdot 3y^2\,dy = \mathbf{E}\,\eta.$$

4. Given variables ξ_1, ξ_2 with finite second moment, show that

$$\mathbf{var}(\xi_1+\xi_2) = \mathbf{var}\,\xi_1 + \mathbf{var}\,\xi_2 + 2\,\mathbf{cov}(\xi_1,\xi_2).$$

Assume that $\xi_1, \xi_2, \ldots$ is a sequence of random variables with $\mathbf{E}\,\xi_k = \mu$ and $\mathbf{var}\,\xi_k = \sigma^2$. They are not independent but there is a constant c such that for every i, $\sum_{k\neq i} \mathbf{cov}(\xi_i,\xi_k) < c$. Using the Chebyshov inequality, prove that the sequence satisfies the Law of Large Numbers:

$S_n/n \to \mu$ in probability.

Solution. Variance of the sum is elementary by expanding the square in $\mathbf{E}[(\xi_1 - \mu) + (\xi_2 - \mu)]^2$. Thus in general, for $S_n = \sum_{i=1}^n \xi_i$, one has

$$\mathbf{var}\, S_n = \mathbf{var}\, \xi_i + \sum_{i \neq k} \mathbf{cov}(\xi_i, \xi_k).$$

We need to show that

$$\mathbf{P}\{|S_n/n - \mu| > \varepsilon\} = \mathbf{P}\{|S_n - n\mu| > n\varepsilon\}$$

converges to 0. By Chebyshov's inequality, the last expression is at most

$$(n\varepsilon)^{-2}\, \mathbf{var}\, S_n = (n\varepsilon)^{-2} \Big[n\sigma^2 + \sum_{i=1}^{n} \sum_{k \neq i} \mathbf{cov}(\xi_i, \xi_k) \Big] \leq \frac{n\sigma^2 + nc}{n^2 \varepsilon^2} \to 0.$$

5. Gamma distribution $\Gamma(n, \lambda)$ is the distribution of a sum of n independent exponentially $\mathrm{Exp}(\lambda)$ distributed random variables. Find

   (a) its characteristic function;

   (b) its mean and its variance;

   (c) For $a > 0$ find the weak limit of the sequence $\{\zeta_n = \xi_n - \sqrt{n}/a\}$, where ξ_n are random variables distributed as $\Gamma(n, a\sqrt{n})$.

Solution. For $\xi_1 \sim \mathrm{Exp}(\lambda)$ the ch.f. is $\varphi_1(t) = \lambda/(\lambda - it)$, with mean $1/\lambda$ and variance $1/\lambda^2$ (e.g., by differentiating the ch.f. at $t = 0$: $\mathbf{E}\, \xi^k = i^k \varphi_1^{(k)}(0)$). Hence for Gamma-distributed ξ, $\varphi_\xi(t) = \lambda^n/(\lambda - it)^n$, $\mathbf{E}\, \xi = -i\varphi_\xi'(0) = n/\lambda$, $\mathbf{var}\, \xi = n/\lambda^2$ as the sum of n independent exponentially distributed r.v.'s. Now, either use CLT for this sum: its mean is $\sqrt{n}/a$ and variance is $1/a^2$, so $a\zeta_n \Rightarrow \mathcal{N}(0, 1)$ implying $\zeta_n \Rightarrow \mathcal{N}(0, a^{-2})$. Or show directly that $\varphi_{\zeta_n}(t) = e^{-it\sqrt{n}/a} a^n n^{n/2}/(a\sqrt{n} - it)^n \to e^{-t^2/(2a^2)}$ for all $t \in \mathbb{R}$.