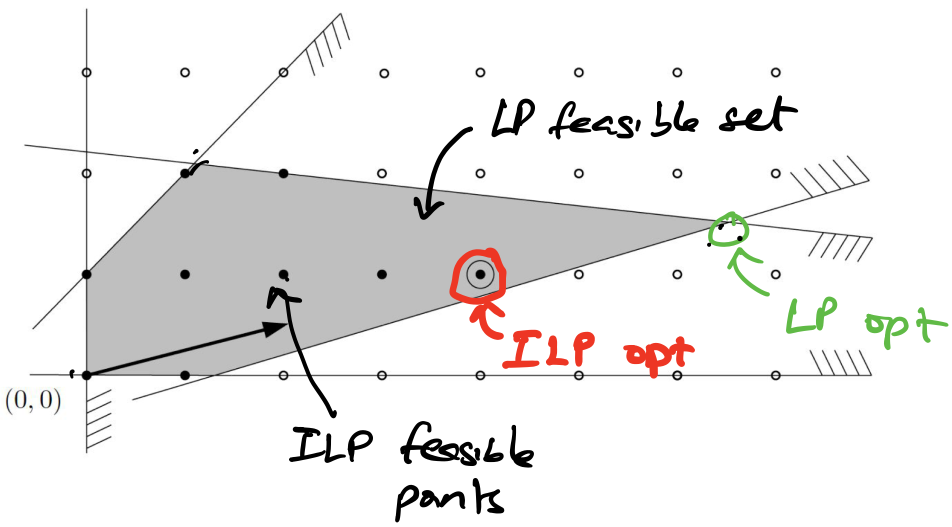
Lecture 4: ILP (cont'd)



ILP		LP (relaxation)
max $c^T x$	$\leq$	max $c^T x$
s.t. $Ax \leq b$		s.t. $Ax \leq b$
$x \geq 0$		$x \geq 0$
integer		
$x \in \{0,1\}$		

Summary

- LP opt $\neq$ ILP opt in general

- LP opt $\geq$ ILP opt (max)

  $\leq$ IPL opt (min)

- Can we use the LP solution to obtain a "good" ILP solution?

Matching

Input: • Bipartite graph

$$G = (U, V, E)$$

• $w: E \to \mathbb{R}^{\geq 0}$

Output: A perfect matching

of maximum weight.

ILP

$$\max \sum_{(i,j)\in E} w_{ij}\, x_{ij}$$

s.t.

$$\sum_{j:\,(i,j)\in E} x_{ij} = 1 \qquad i \in U$$

$$\sum_{i:\,(i,j)\in E} x_{ij} = 1 \qquad j \in V$$

$$x_{ij} \in \{0,1\} \qquad (i,j)\in E.$$

// $x_{ij} = 1$ if (i,j) included in matching

Observe:

ILP is an exact formulation of the perfect matching problem.

$$\left\{ \begin{array}{c} \text{perfect matchings} \\ \text{in } G \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{feasible solutions} \\ \text{to the ILP} \end{array} \right\}$$

- If M is a perfect matching in G,

$$x_{ij} = \begin{cases} 1 & (i,j) \in M \\ 0 & \text{o.w.} \end{cases}$$

will be a feasible solution to ILP.

- If $\{\tilde{x}_{ij}\}$ a feasible solution to ILP

$$M = \{(i,j) \in E \mid x_{ij} = 1\}$$

is a perfect matching in G.

Pass to the LP relaxation:

LP relaxation

$$\max \sum_{(i,j) \in E} w_{ij} x_{ij}$$

s.t.

$$\sum_{\substack{j \\ (i,j) \in E}} x_{ij} = 1 \qquad i \in U$$

$$\sum_{\substack{(i,j) \in E \\ i}} x_{ij} = 1 \qquad j \in V$$

$$x_{ij} \geq 0 \qquad (i,j) \in E$$

LP opt = ILP opt !

& LP opt is integral !

Total Unimodularity

A matrix A is **totally unimodular (TUM)** if every square submatrix of A has determinant $\in \{-1, 0, 1\}$

Examples:

- $\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$ ✗

TUM matrices must have entries $\in \{-1, 0, 1\}$

- $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ ✗ $\det = 2$

$$\begin{array}{cc} & \begin{array}{ccc} \downarrow & & \downarrow \end{array} \\ \begin{array}{c} \rightarrow \\ \\ \rightarrow \end{array} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \end{array}$$ ✗

- $\begin{bmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ \hline 0 & 1 & 0 \end{bmatrix}$ ✓

Condition for TUM. (Sufficient)

A matrix A is TUM if

- entries are $\in \{-1, 0, 1\}$

- rows can be partitioned into two sets so that in each column, there are at most 2 non-zero entries and
  - if the 2 non-zero entries are of the same sign, they are in opposite sides of the partition
  - if the 2 non-zero entries are of different sign, then they are on the same side of the partition.

Hoffmen-Kruskal Theorem.

The vertices of the region

$\{x : Ax \leq b\}$

are integral if A is TUM and b is integral.

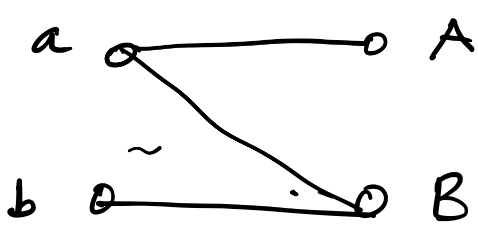
Summary:

- can formulate problem exactly as ILPs (bipartite matching)
- ILPs are hard, so we pass to the LP relaxation.
- Hoffman-Kruskal Theorem tells us that the LP opt is integral if the coeff matrix A is <u>TUM</u> & rhs is integral.

LP opt = ILP opt

Is the coeff. matrix in the bipartite maching problem TUM?

Example:



x_{aA}, x_{aB}, x_{bB}

$$x_{aA} + x_{aB} = 1 \qquad (a)$$

$$x_{b,B} = 1 \qquad (b)$$

$$x_{a,A} = 1 \qquad (A)$$

$$x_{aB} + x_{bB} = 1 \qquad (B)$$

$$\begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} x_{aA} \\ x_{aB} \\ x_{bB} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\underbrace{\qquad\qquad}_{?}$$

$$\begin{array}{c} \\ a \\ b \\ A \\ B \end{array}\begin{array}{c} \begin{array}{ccc} aA & aB & bB \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \begin{bmatrix} x_{aA} \\ x_{aB} \\ x_{bB} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

What is this matrix called?

- Node-edge incidence matrix

(• vertex-vertex incidence matrix)

A is the node-edge incidence matrix of a graph G if

- the rows correspond to vertices
- the columns " " edges

$$A(i,j) = \begin{cases} 1 & \text{edge } j \text{ incident on vertex } i \\ 0 & \text{o.w.} \end{cases}$$

Is the node-edge incidence matrix of a bipartite graph TUM?

Use the criteria we discussed — try to partition the rows!

To partition the rows, use the fact that the graph is bipartite! Partition according to the bipartition & observe that in a bipartite graph all edges go from one side to the other, hence the two 1s in each column will be on opposite sides of the partition.

Proposition: The node-edge incidence matrix of a bipartite graph is

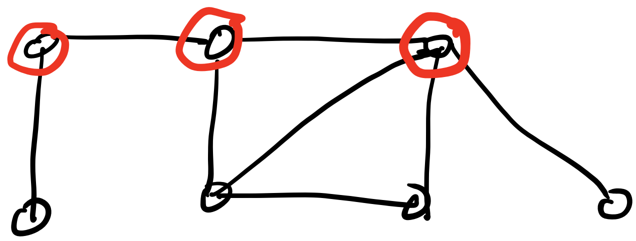
TUM.

Corollary: ILP = LP for bipartite matching!

Observations

- Bipartite matching can thus be solved in polynomial time.
- There are other algorithms to solve bipartite matching (Hungarian method)
- LP is very simple to use here.
- Matching is also polynomial time solvable for general i.e. non-bipartite graphs. But then this LP-alg. doesn't work.

Vertex Cover.



$G=(V,E)$

Vertex cover is a subset $U \subseteq V$

s.t. for every edge $(u,v) \in E$,

either $u \in U$ or $v \in U$.

Find a vertex cover of minimum size.

ILP exact formulation.

$$x_i \in \{0,1\} \quad i \in V$$

$$x_i = \begin{cases} 1 & i \in \text{vertex cover} \\ 0 & \text{o.w.} \end{cases}$$

ILP:

$$\min \sum_i x_i$$

$$x_i + x_j \geqslant 1 \qquad (i,j) \in E$$

$$x_i \in \{0,1\}$$

Exact formulation of vertex-cover

$$\left\{ \begin{array}{c} \text{vertex covers} \\ \text{in } G \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} \text{integer } 0/1 \\ \text{solutions to ILP} \end{array} \right\}$$

- U is a vertex cover then

$$\tilde{x}_i = \begin{cases} 1 & i \in U \\ 0 & \text{o.w.} \end{cases}$$

is a solution to ILP.

- If $\{\tilde{x}_i\}$ are a solution to ILP

$$U = \{ i : x_i = 1 \}$$

is a vertex cover.

So, should we pass to the LP relaxation (ignore 0/1 constraints) and hope that we will get integral solutions anyway?

Warning! Vertex Cover is NP-hard.

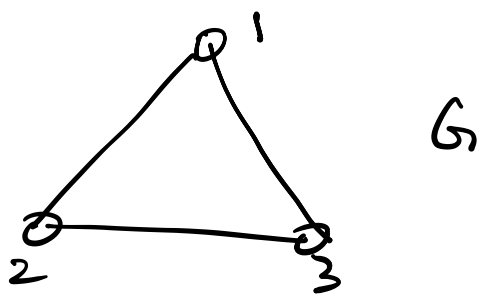
We have just shown a <u>reduction</u> from vertex cover to ILP.

<u>Theorem</u>

ILP is NP-hard!

→ cannot expect to solve ILP in poly time !

Example:



- min size vertex cover ?

  2

- ILP optimum = 2

  $\neq$

- LP optimum = 1.5

  $x_1^* = x_2^* = x_3^* = 0.5$

$$\min \quad x_1 + x_2 + x_3$$

$$\text{s.t.} \quad x_2 + x_3 \geq 1$$

$$x_3 + x_1 \geq 1$$

$$x_1 + x_2 \geq 1$$

$$x_1, x_2, x_3 \geq 0$$

Summary

- In bipartite matching the special structure meant that

$$\text{LP opt} = \text{ILP opt}$$

- In general,

$$\text{LP opt} \neq \text{ILP opt}$$

& we need to do something extra using LP solutions.