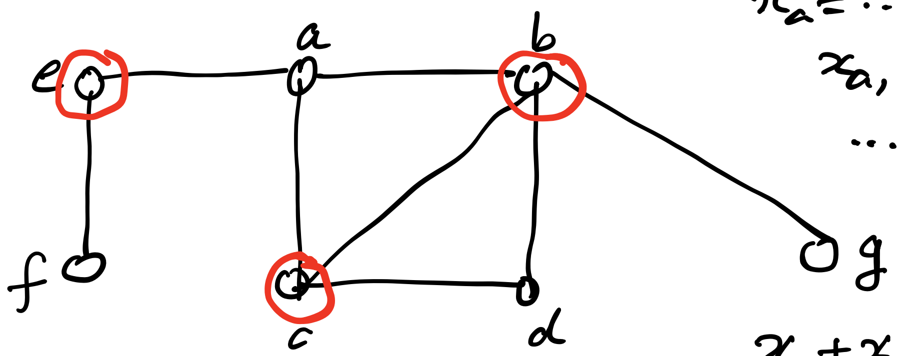
Lecture 5: ILP, Rounding & Branch Bound

Vertex Cover

$x_b = 1,\ x_c = 1,\ x_e = 1$

$x_a = \cdots = x_g = 0$

$x_a, x_b \ldots x_g$



$x_a + x_b \geq 1$

A vertex cover is a subset $U \subseteq V$ s.t. every edge is "covered" i.e. for every edge $(i,j) \in E$, either $i \in U$ or $j \in U$.

Find a vertex cover of minimum size.

Input:

- Graph $G = (V, E)$
- $c : V \to \mathbb{R}^{\geq 0}$

Output : a vertex cover $U \subseteq V$ of minimum cost i.e.

$\min \sum_{i \in U} c_i$ s.t. for every edge $(i,j) \in E$, $i \in U$ or $j \in U$.

$x_i \in V$, $x_i \in \{0, 1\}$

$$x_i = \begin{cases} 1 & \text{if } i \in U \\ 0 & \text{o.w.} \end{cases}$$

ILP: $G = (V, E)$, $c : V \to \mathbb{R}^{\geq 0}$

$$\min \sum_{i \in V} c_i x_i$$

s.t. $x_i + x_j \geq 1 \quad (i,j) \in E$

$x_i \in \{0, 1\} \quad i \in V$

Claim: Exact formulation of the VC problem!

$$\left\{ \begin{array}{c} \text{feasible solutions} \\ \text{to ILP} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{vertex} \\ \text{covers in} \\ G \end{array} \right\}$$

- If U is a vertex cover, then feasible solution to ILP is given by:

$$x_i = \begin{cases} 1 & i \in U \\ 0 & \text{o.w.} \end{cases}$$

This satisfies all constraints of ILP if $(i,j) \in E$, $i \in U$ or $j \in U$ i.e. $x_i = 1$ or $x_j = 1$ & hence $x_i + x_j \geq 1$.

- If $\{\tilde{x}_i\}$ is a feasible sol to ILP, then vertex cover U is given by:

$$U = \{ i : \tilde{x}_i = 1 \}$$

is a vertex cover. Because if $(i,j) \in E$, $\tilde{x}_i + \tilde{x}_j \geq 1$ which means $\tilde{x}_i = 1$ or $\tilde{x}_j = 1$, and hence $i \in U$ or $j \in U$ (or both). i.e. edge (i,j) is covered.

Pass to the LP-relaxation!

$$\min \sum_{i \in V} c_i x_i$$

s.t.

$$x_i + x_j \geq 1 \qquad (i,j) \in E$$

$$x_i \geq 0 \qquad i \in V$$

However, output from LP-solver could be fractional e.g. $x_1 = 0.62$
$x_2 = 0.38 \ldots$

Sometimes there is special structure in LP problem that the opt sol is integral. e.g.

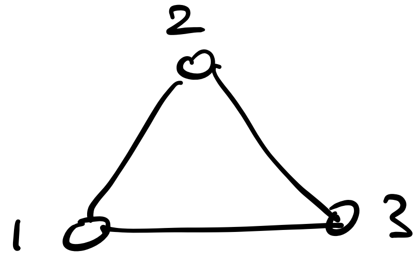
Coefficient matrix is TUM

$\longrightarrow$ integrality of solution.

In (bipartite) matching, the coefficient matrix was: node-edge incidence matrix:

$$A \quad \begin{array}{cc} & \text{columns } e \in E \\ \begin{array}{c}\text{node } i \\ i \in V\end{array} & \left[\begin{array}{c} \vdots \\ \cdots \; - \; \circ \\ \end{array}\right] \end{array} \qquad A(i,e) = \begin{cases} 1 & i \in e \\ 0. & \text{o.w.} \end{cases}$$

Example



$x_1^* = x_2^* = x_3^*$
$= 0.5$

$$\min \quad x_1 + x_2 + x_3$$

s.t.

$$x_2 + x_3 \geq 1$$
$$x_3 + x_1 \geq 1$$
$$x_1 + x_2 \geq 1$$
$$x_1 \ldots x_3 \geq 0$$

$$\begin{array}{cc} & \overset{\text{vertices}}{\begin{array}{ccc} 1 & 2 & 3 \end{array}} \\ \text{edges} \begin{array}{c} (2,3) \\ (3,1) \\ (1,2) \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Transpose of the coeff matrix in matching problem!

A is TUM $\Leftrightarrow$ A^T TUM

The coeff matrix is TUM if

the graph is <u>bipartite</u>

In general, graph not bipartite,
the coeff matrix not guaranteed TUM
hence, opt. LP sol not alway integral.

If G is bipartite, then opt is
integral because coeff matrix is TUM
& hence vertex cover in bipartite
graphs can be solved by LP.

So in a general graph, the LP
optimum x^* will be fractional.
We will need to round it.

Rounding rule:

$$\tilde{x}_i = \begin{cases} 1 & x_i^* \geq 1/2 \\ 0 & \text{o.w.} \end{cases}$$

Questions:

- Correctness: does this yield a vertex cover?
- Quality: How good is the vertex cover i.e. how far from optimal solution?

$$U := \{ i \in V : \tilde{x}_i = 1 \}$$

Is U a vertex cover?

i.e. for every edge $(i,j) \in E$, $\tilde{x}_i = 1$ or $\tilde{x}_j = 1$?

Yes, U is a vertex cover!

Why? Let's look at any edge $(i,j) \in E$

$$x_i^* + x_j^* \geq 1$$

$\rightarrow$ $x_i^* \geq 1/2$ or $x_j^* \geq 1/2$

$\rightarrow$ $\tilde{x}_i = 1$ or $\tilde{x}_j = 1$

$\rightarrow$ $i \in U$ or $j \in U$

i.e. U is a vertex cover.

How close to optimal is U?

$$\text{cost}(U) = \sum_{i \in U} c_i$$

$$= \sum_{i \in V} c_i \tilde{x}_i$$

$$\leq 2 \underbrace{\sum_{i \in V} c_i x_i^*}_{\text{LP opt.}}$$

Since $\tilde{x}_i \leq 2 \cdot x_i^*$

if $x_i^* < 1/2$, $\tilde{x}_i = 0$

if $x_i^* \geq 1/2$ $\tilde{x}_i = 1$

$$= 2 \cdot \text{LP opt.}$$

$$\leq 2 \cdot \text{ILP-opt.}$$

$$= 2 \cdot (\text{opt vertex cover})$$

Our algorithm always delivers a vertex cover whose cost is at most twice the cost of of the optimal vertex cover.

i.e. our algorithm is an

approximation algorithm for vertex cover in general graphs (where VC is known to be NP-hard) which always delivers a solution whose **approximation factor** is guaranteed to be at most 2.

i.e.
$$\frac{\text{cost of solution delivered by algorithm}}{\text{cost of opt sol.}} \le 2$$

i.e. we have a **2-approx** algorithm that runs in polynomial time.

This is best possible in theory!

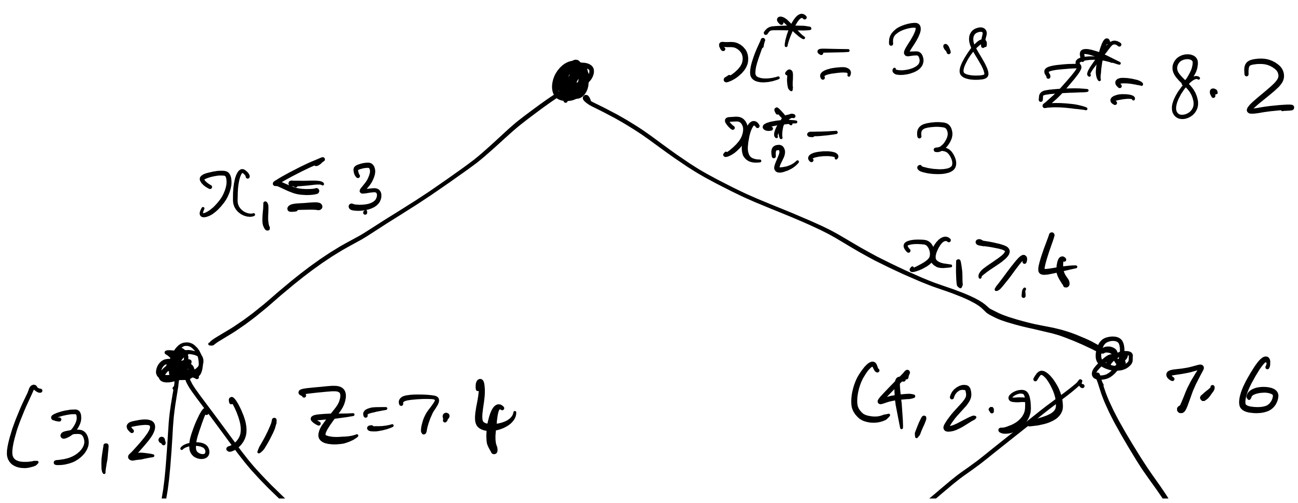
No algorithm can achieve a better approx factor than 2 in poly time.

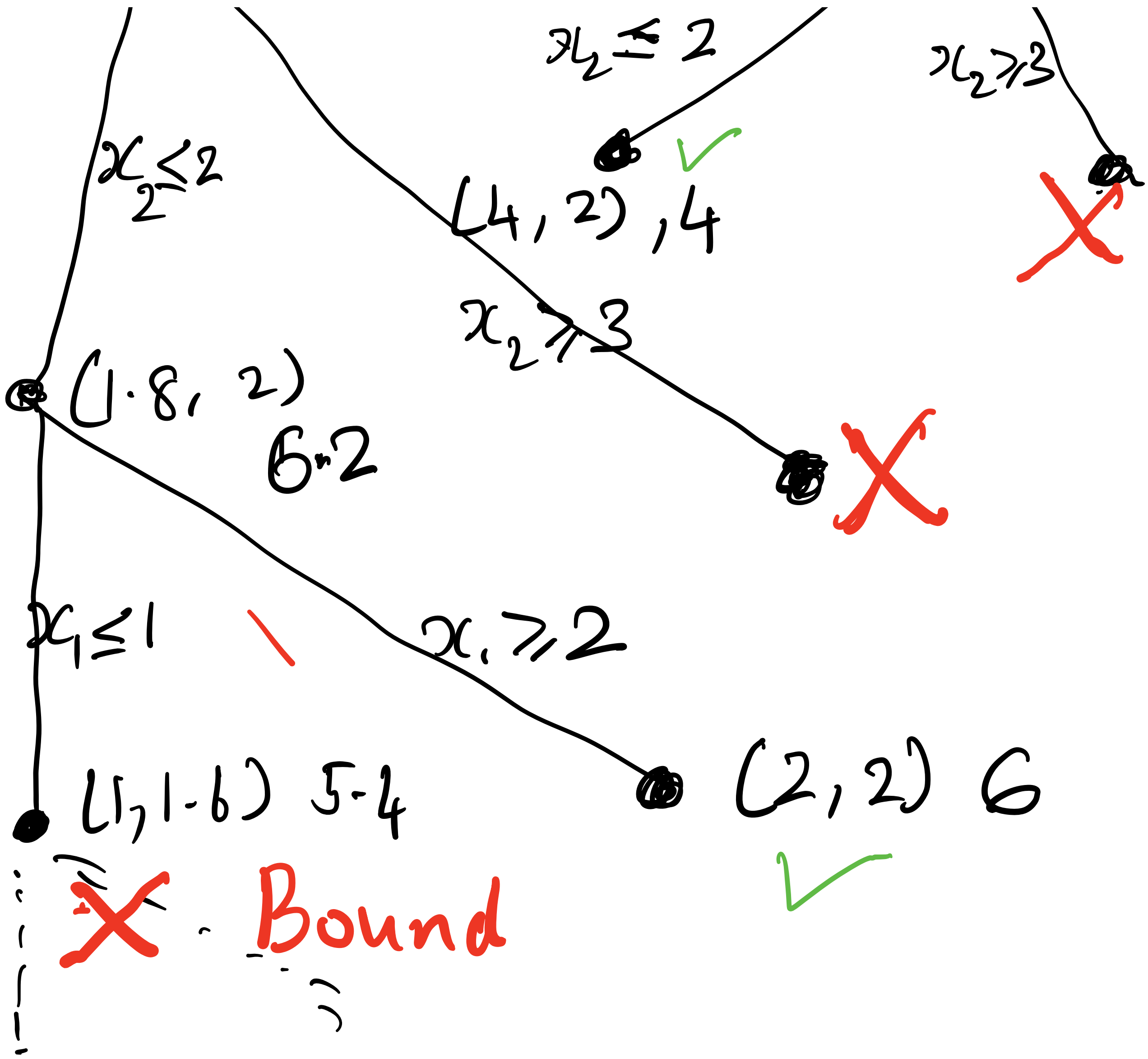
(Johan Håstad "PCP Theorem")

Suppose we do want to solve ILP exactly. Branch & Bound!

$$\boxed{\begin{array}{ll} \tilde{x}_1 = 2 & \tilde{z} = 6 \\ \tilde{x}_2 = 2 & \end{array}}$$

opt





$x_2 \le 2$
$x_2 \ge 3$
$x_2 \le 2$
(4, 2), 4
$x_2 \ge 3$
(1.8, 2)
6.2
$x_1 \le 1$
$x_1 \ge 2$
(1, 1.6) 5.4
(2, 2) 6
Bound