Lecture 10:

Algebra & Geometry of LP

Standard Forms of LP

$$\max \; c^T x \qquad\qquad \min \; b^T y$$

$$Ax \leq b \qquad\qquad A^T y \geq c$$

$$x \geq 0 \qquad\qquad y \geq 0$$

Example

$$\max \; 3x_1 - 2x_2$$

$$2x_1 - x_2 \leq 4$$

$$x_1 + 3x_2 \geq 5$$

$$x_2 \geq 0$$

Can be rewritten in Equational Standard

_____ Form

$$\max \ c^T x$$

s.t.

$$Ax = b$$

$$x \geq 0$$

(1) $2x_1 - x_2 \leq 4$

$$\Updownarrow$$

$$2x_1 - x_2 + x_3 = 4$$

slack variable

$$x_3 \geq 0$$

(2) $x_1 + 3x_2 \geq 5$

$$\Updownarrow$$

$$x_1 + 3x_2 - x_4 = 5$$

slack variable

$$x_4 \geq 0$$

(3) x_1 unrestricted

$$\Updownarrow$$

$$x_1 = x_5 - x_6$$

$$x_5, x_6 \geq 0$$

Original LP is equivalent to:

$$\max \ 3x_5 - 2x_2 - 3x_6$$

$$\text{s.t.} \ 2x_5 - 2x_6 - x_2 + x_3 = 4$$

$$x_5 - x_6 + 3x_2 - x_4 = 5$$

$$x_2, x_3, x_4, x_5, x_6 \geq 0$$

We can always transform <u>any</u> LP into an equivalent LP in equational form.

Algebra of LP in equational form.

$$\max \ c^T x$$

$$Ax = b$$

$x \geq 0$ ← not present in linear alg.

Assumptions

If A is a $m \times n$ matrix

then can assume $m \leq n$.

- If $m > n$ →

either $\begin{cases} \text{no solution} \\ \text{some equations are redundant} \end{cases}$

Basic Feasible Solutions.

A basic feasible solution to

$$\max c^T x \quad \text{s.t} \quad Ax = b, \quad x \geq 0$$

is a feasible solution $x \in \mathbb{R}^n$ (i.e. $Ax = b$) s.t. there exist a subset $B \subseteq \{1, .., n\}$ of columns of size m, $|B| = m$ s.t.

- the square submatrix A_B (i.e. the submatrix of columns in B) is non-singular i.e. the columns indexed by B are linearly ind.

- $x_j = 0 \quad j \notin B$.

Example.

$$A = \begin{array}{c} \\ \\ \end{array}\begin{array}{c} \quad\quad\quad\quad B \\ \left(\begin{array}{ccccc} 1 & 5 & 3 & 4 & 6 \\ 0 & 1 & 3 & 5 & 6 \end{array}\right) \end{array} \quad \begin{array}{l} m=2 \\ n=5 \end{array}$$

$b = (14, 7)$

$B = \{2, 4\}$

$$A_B = \begin{pmatrix} 5 & 4 \\ 1 & 5 \end{pmatrix} \qquad \det A_B \neq 0$$

$$x = (0, 2, 0, 1, 0) \qquad , x \geq 0$$

Claim: x is a BFS.

- $Ax = b, \qquad x \geq 0$

- $x_1 = 0,\ x_3 = 0,\ x_5 = 0$

Theorem

For any LP in equational form

$$\max\ c^T x \quad \text{s.t. } Ax = b,\ x \geq 0$$

which is feasible, then there is always a BFS that is optimal.

If x is a BFS corresponding to columns B,

$$x = (x_B, x_{\bar{B}})$$

$$A = (A_B, A_{\bar{B}})$$

$$Ax = A_B x_B + A_{\bar{B}} \cdot \underbrace{x_{\bar{B}}}_{0} = b$$

$$A_B x_B = b$$

$$x_B = A_B^{-1} b$$

& this will be the optimal sol to LP

provided that $x_B \geq 0$.

Suggests an algorithm to solve LP!

Optimum always exist as a BFS

If a LP (in equational form) is feasible, then there is always an optimal solution which is a BFS.

Algorithm

Search among all BFS for opt.

max $\leftarrow -\infty$

for all $|B| = m$, $B \subseteq \{1, \dots, n\}$

- check if A_B is non-singular
- if yes, solve (restricted to colums in B)

$$A_B x_B = b_B$$
$$x_B = A_B^{-1} b_B$$

- check if $x_B \geq 0$

$$x_j = 0 \quad \text{for } j \notin B$$

- if $c^T x > \max$, $\max \leftarrow c^T x$

Running time will be at least as much as number of iterations

$$\binom{n}{m}$$

Think about $n = 2m$

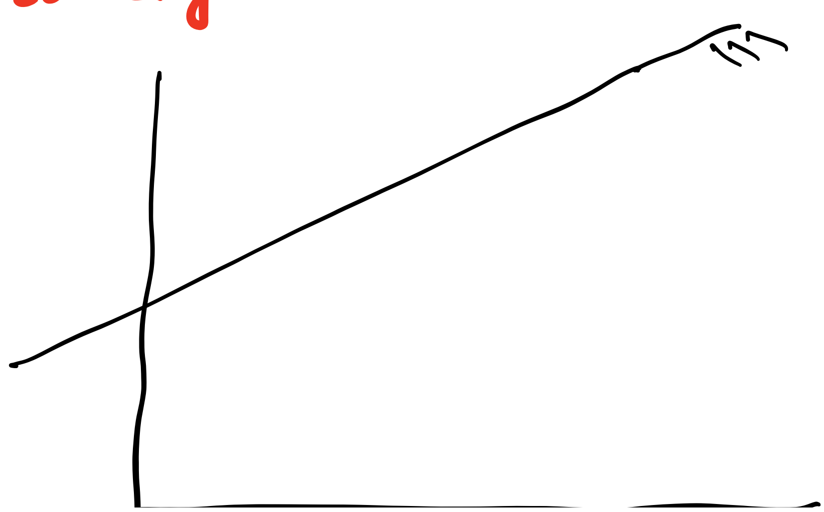
$$\binom{2m}{m} \sim \geq 2^m$$

→ exponential time alg.

Not a good algorithm. Need a smartmer way to search among all BFS.

Simplex is a search through all BFS, but in a smart way!

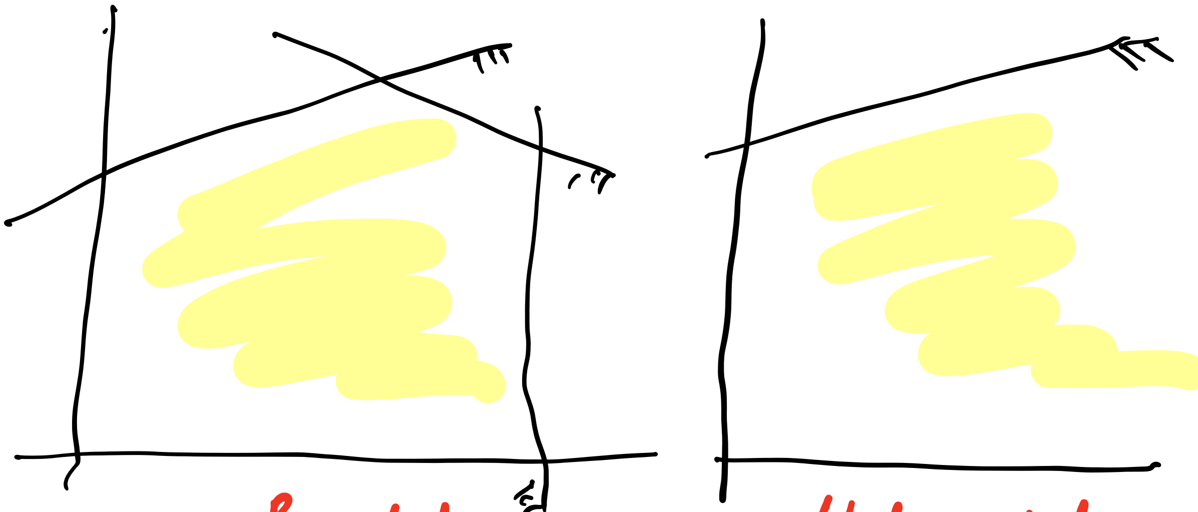
Geometry.



Halfspaces

$$H = \{ x \in \mathbb{R}^n \mid a^T x \geq b \}$$

Feasible set of LP



Bounded

Unbounded

$\bigcap_{i=1}^{m} H_i$

Polytope

Polyhedron

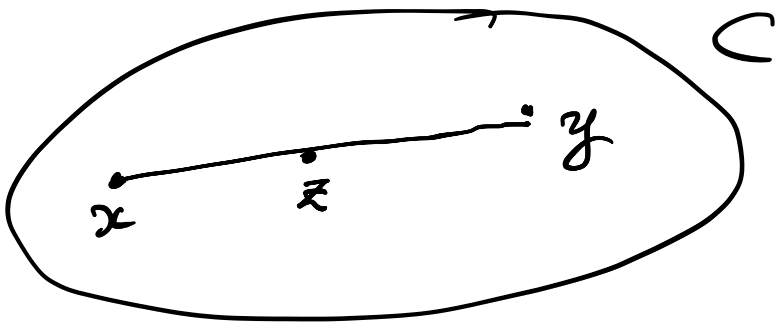
Convexity

A subset $C \subseteq \mathbb{R}^n$ is

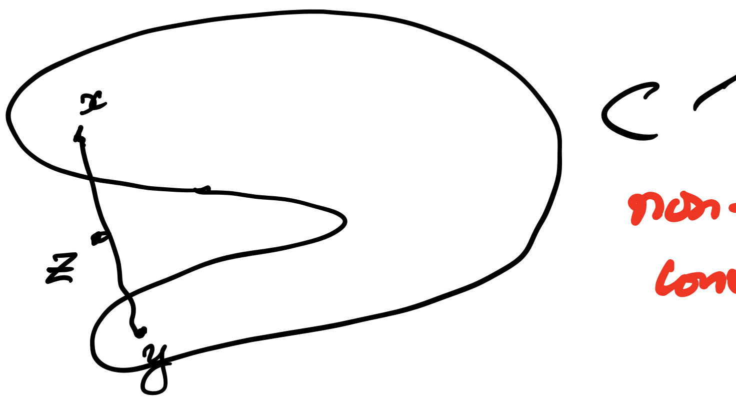
Convex if for all $x, y \in C$

$$z := \lambda x + (1-\lambda) y \in C$$

for all $0 \le \lambda \le 1$



Convex



non-convex

Check:

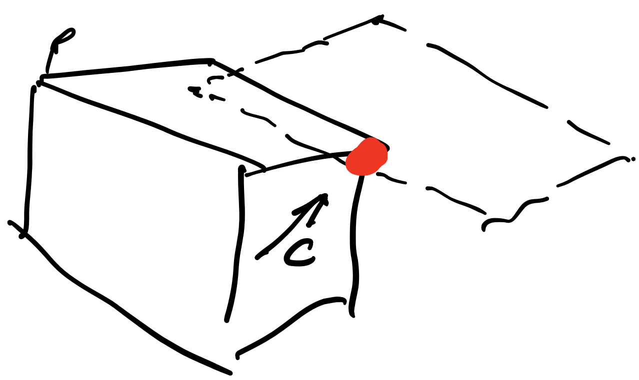
- half space is convex
- finite intersection of convex sets is again convex

→ Feasible set of LP is convex

What does the optimal solution to an LP look like?

A <u>vertex</u> of a polyhedron P is a point $x \in P$ s.t. there is a vector $c \in \mathbb{R}^n$ with

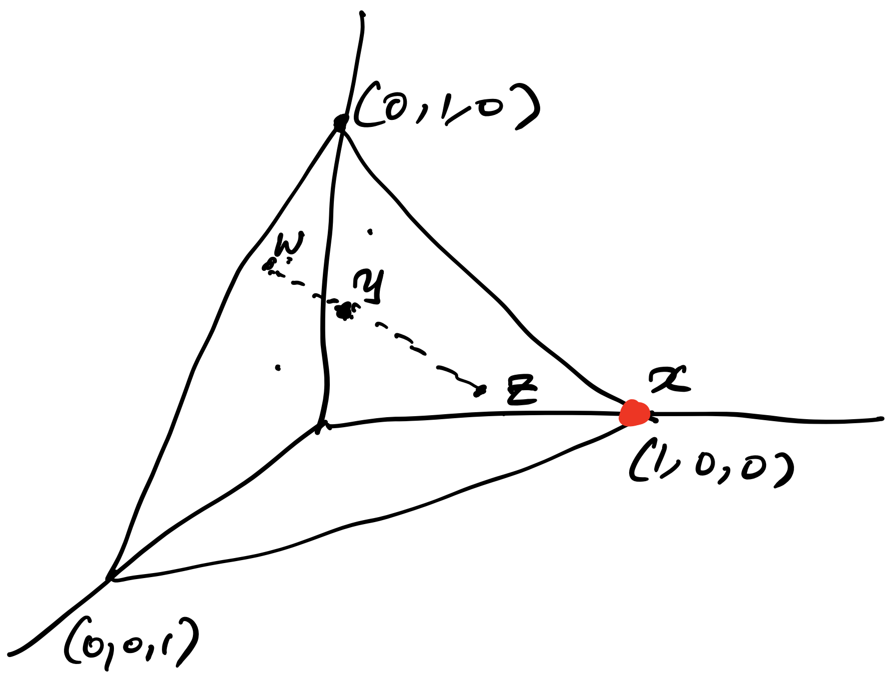
$$c^T x > c^T x' \quad \text{for all } x' \in P.$$



Equivalently, a vertex is a point $x \in P$ s.t. for no other points $y, z \in P$ is

$$x = \lambda y + (1-\lambda) z, \quad \begin{matrix} y \neq x \\ z \neq x \end{matrix}$$

for any $0 \le \lambda \le 1$.



Optimum of a LP is always achieved at a vertex!

Algebra $\qquad\qquad$ Geometry

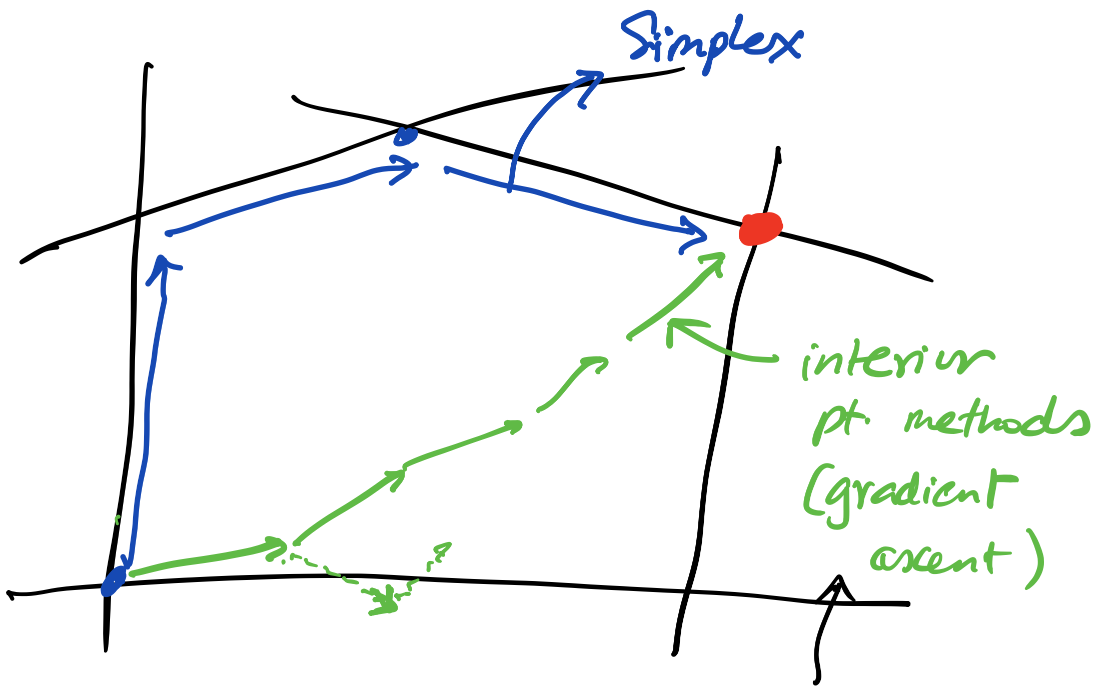
$\{\text{BFS}\} \longleftrightarrow \{\text{vertices}\}$

Algorithm for a LP can always restrict search to vertices of the feasible set.

But there can be an exponential number of vertices of a polyhedron defined by a n inequalities over $2n$ variables.

Simplex is a search over vertices of a polyhedron (feasible set of LP) but in a smart way, not exhaustive.

However, other algorithms do go through interior of polygon to get to optimal.



Boyd & Vandenberghe
"Convex Optimization"