TDA206/DIT370: Discrete Optimization Final Exam

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Instructions:

- Write your answers to the point. Credit will only be assigned based on the correctness of what is requested e.g. if it is a LP, credit will be assigned based on the choice of decision variables, objective and constraints only. Anything else you write will not receive any credit. So use extra sheets to work out the answer and then only write the requested answer in your solution.
- You may use the textbook and all other reading material mentioned in the reading with the lectures on the course page.

1. (10 points) A company produces four electronic components A,B,C and D. The components can be produced in any of three machines, each of which has a given capacity. The table below gives how many of each component a machine can produce in one hour:

	1	2	3
A	300	600	800
B	250	400	700
C	200	350	600
D	150	200	300

Suppose each machine can be used at most 50 hours each week. The hourly cost of each machine is 300, 500 and 800 kr, respectively. The company must produce each week, 7000, 8000, 5000 and 6000 units of each component respectively. Formulate a LP to produce the components at minimum cost.

2. (10 points) We consider the vertex cover problem again, but now we're allowed to compute a *partial cover* i.e. some edges may not be covered but then we need to pay for them separately. More precisely, the input is an undirected graph $G = (V, E)$ together with costs $w_i \geq 0, i \in V$ on the vertices and a separate set of costs $c_{i,j} \geq 0, (i, j) \in E$ on the edges. If we select the subset $U \subseteq V$, then the cost we pay is:

$$\sum_{i \in U} w_i + \sum_{\substack{(i,j) \in E, \\ i \notin U, j \notin U}} c_{i,j}.$$

The first sum is the cost payed for the selected vertices in U, and the second sum is for the edges that are not covered by U. The problem is to choose U such that this total cost is minimized.

(a) Give an exact formulation of the problem as an ILP.
Solution Introduce decision variables $y_{i,j} \in \{0, 1\}$ such that $y_{i,j} = 1$ iff (i, j) is not covered:

$$\begin{aligned} & \max \sum_{i \in V} w_i x_i + \sum_{(i,j) \in E} c_{i,j} y_{i,j} \\ \text{s.t.} \quad & x_i + x_j + y_{i,j} \geq 1, \quad (i, j) \in E \\ & x_i, y_{i,j} \in \{0, 1\} \end{aligned}$$

(b) Pass to the LP relaxation (state the changes to the ILP) and give an algorithm to select U based on rounding the optimal solution to the LP. Give the rounding rule explicitly and argue why it is a feasible solution to the ILP.

Solution: Replace integer constraints by $x_i, y_{i,j} \geq 0$. Round $\hat{x}_i = 1$ if $x_i^* \geq 1/3$, $\hat{y}_{i,j} = 1$ if $y_{i,j}^* \geq 1/3$. The constraint $x_i + x_j + y_{i,j} \geq 1$ guarantees that at least one of the three variables is rounded to 1, hence $(\hat{x}, \hat{y})$ is a feasible solution.

(c) Analyse the quality of the resulting approximation algorithm and show it is a constant factor approximation: give the approximation factor and prove that the algorithm achieves this approximation.

Solution: Since $\hat{x}_i \leq 3x_i^*$ and $\hat{y}_{i,j} \leq 3y_{i,j}^*$, the algorithm delivers a 3 approximation.

3. (10 points) Consider the LP:

$$\begin{array}{rrl} \max & z = & 2x_1 + 3x_2 + x_3 \\ & \text{s.t.} & x_1 - x_2 + 2x_3 \leq 1 \\ & & 4x_1 + 2x_2 - x_3 \leq 2 \\ & & x_1, x_2, x_3 \geq 0 \end{array}$$

(a) Write the dual LP.

(b) Determine if the dual solution $y_1 = 5/3$ and $y_2 = 7/3$ is optimal using complementary slackness.

4. (10 points) Consider again the vertex cover problem, this time in hypergraphs. Consider the hypergraph with:

- vertex set: $V := \{A, B, C, D, E, F\}$
- hyperedges: $e_1 := \{A, B, E\}$, $e_2 := \{B, C, E\}$ and $e_3 := \{C, D, F\}$.
- weights: $w_A := 3, w_B := 2, w_C := 1, w_D := 4, w_E := 2, w_F := 3$.

(a) Write the LP relaxation of the min cost vertex cover problem for this particular hypergraph i.e. to choose the min cost set of vertices that cover all three hyperedges (i.e. at least one vertex in each of the three hyperedges must be chosen). (Your LP must have six variables and 3 constraints.)

(b) Write the dual.

(c) Apply the primal dual algorithm raising dual variables as much as possible until dual constraints go tight. Show the variable values at each iteration and show the final solution, comparing it to the optimal solution.

5. (10 points) Let $X = [-a, a] \subseteq R$ (for some $a > 0$) and consider the function $f(x) := x^4$.

(a) Prove that is smooth over X and determine a specific smoothness parameter L (depending on a). (You may use the fact that f is smooth with paramter L if $|f'(y) - f'(x)| \leq L|y - x|$.

(b) Give the convergence rate for gradient descent applied to minimize f starting from a point in X.