Computational Methods for SDEs

Introduction and information

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Goal

Compute *efficiently* and *accurately*

expectation

random variable

$$\mathbb{E}[\varphi(Y)]$$

special RVs

UQ

uncertainty quantification

functional

quantity of interest

QoI

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Compute *efficiently* and *accurately*

$$\mathbb{E}[\varphi(Y)]$$

e.g., Y is given by solution to

$$\begin{aligned} \mathrm{d}X_t &= b(t, X_t)\ \mathrm{d}t + \sigma(t, X_t)\ \mathrm{d}W_t \\ X_0 &= x \end{aligned}$$

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- How do we approximate solutions?
- How good are theses approximations?
- How are we “sufficiently” efficient?

Approach

- Define our expectations of the course

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- Theoretical derivations
  - approximation
  - convergence
  - [regularity]

analysis

math

functional

numeric

stochastic

Approach

- Define our expectations of the course
- Theoretical derivations
  - approximation
  - convergence
  - [regularity]
- Computer simulations to test the theoretical findings
  $\longrightarrow$ How do things work in practice?

About the course

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- Your chance (due to small group): give input and shape
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  - Come with parts and questions that we should discuss.
  - Dare to say what you do not understand.
  - If time admits, I pick out parts that are important to discuss from my point of view.
  - I will *not* go through the text/proofs if not explicitly requested.

About me — About you

- Why are you here?
- What do you expect?
- What is your background?

Formalities

- *officially*: 4h exam, 2 projects with bonus points
- *teaching*: 4h lectures + 2 h exercise classes (Per Ljung)
  - *lectures*: Monday + Thursday, 10 - 12 in MVF33
  - *exercise classes*: Tuesday, 10 - 12 in MVF26
  - *exceptions*: *tba thanks to the pandemic*

Back to mathematics

Monte-Carlo

multi-level MC

understand solution

Euler-Maruyama

Milstein

strong and weak convergence

$$\underbrace{\mathbb{E}[\varphi(X_t)]}$$

Feynman-Kac

Kolmogorov equations

(PDE)

finite element approximations

Literature

[G] Emmanuel Gobet: Monte-Carlo Methods and Stochastic Processes: From Linear to Non-Linear, CRC, 2016

[HRSW] Norbert Hilber, Oleg Reichmann, Christoph Schwab, Christoph Winter: Computational Methods for Quantitative Finance: Finite Element Methods for Derivative Pricing, Springer, 2013

[KP] Peter Kloeden, Eckhard Platen: Numerical Solutions of Stochastic Differential Equations, Springer, 1992

[Ø] Bernt Øksendal: Stochastic Differential Equations: An Introduction with Applications, Springer, 2003

Content

- Chapter 4–6 in [G]
- Chapter 3, 4, 8, 9 in [HRSW]

In words

- Review of Brownian motion, Itô integration, SDEs
- Feynman–Kac formulas
- Euler–Maruyama scheme, strong & weak convergence
- Statistical errors, (multilevel) Monte Carlo methods
- Review on FEM for parabolic PDEs
- FEM methods for PDEs from the Feynman–Kac formulas
- If time and interest: Applications

Recall on probability theory

- $(\Omega, \mathcal{A}, P)$ *probability space*
  - Ω set
  - $\mathcal{A}$ σ-algebra
  - P probability measure
- $X : \Omega \to \mathbb{R}$ *random variable*
  - $\mathcal{B}(\mathbb{R}) = \sigma([a, b), a < b)$ Borel σ-algebra
  - X is $\mathcal{A}/\mathcal{B}(\mathbb{R})$-measurable, i.e.
    $\forall B \in \mathcal{B}(\mathbb{R}) : \{\omega \in \Omega, X(\omega) \in B\} \in \mathcal{A}$

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- $P_X(B) = P(\{\omega \in \Omega, X(\omega) \in B\})$ *image measure*, $B \in \mathcal{B}(\mathbb{R})$
- f *density* of X

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- $\mathbb{E}[X]$ *expectation* of X with

$$\mathbb{E}[X] = \int_\Omega X \, \mathrm{d}P = \int_{-\infty}^{\infty} x f(x) \, \mathrm{d}x$$

Random numbers

Do random numbers exist?

- philosophical question
- USB device uses *Johnson–Nyquist noise*
- *HERE* generation of *pseudo random numbers*

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Definition

- $U = (U^{(i)}, i \in \mathbb{N})$ sequence of independent, identically distributed random variables uniformly on $[0, 1)$
- *pseudo random number*:
  - sequence of numbers
  - generated by a (deterministic) algorithm
  - behaves like U

Random number generators

generate on $[0, 1)$ uniformly distributed random numbers

- *Algorithm K* of Knuth
- *linear congruent pseudo random number generator* of Lehmer

$$X_0 = \text{“seed”}$$
$$X_{n+1} = aX_n + c \bmod m$$

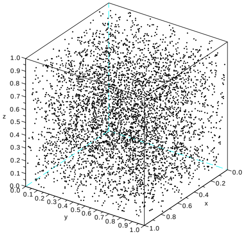
imitates roulette, output:

$$\frac{X_n}{m}$$

  - *RANDU* by IBM ($m = 2^{31}$, $a = 2^{16} + 3$, $c = 0$)
  - *Mother* by Marsaglia
- *Mersenne Twister* by Matsumoto & Nishimura
- *KISS* (Keep It Simple, Stupid) by Marsaglia & Zaman

Test of RANDU

- generate random numbers $(U^{(i)}, i \in \mathbb{N})$, $U^{(i)} \sim \mathcal{U}([0,1))$
- set triplets $(U^{(1)}, U^{(2)}, U^{(3)})$, $(U^{(4)}, U^{(5)}, U^{(6)})$, etc.
- draw them into the cube $[0,1] \times [0,1] \times [0,1]$



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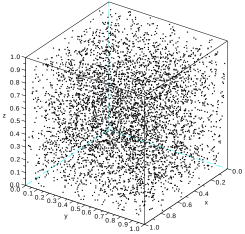

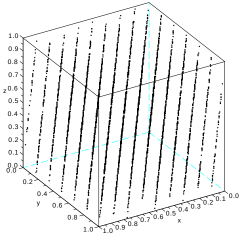

Random numbers for given distributions

- general methods
  - inversion method

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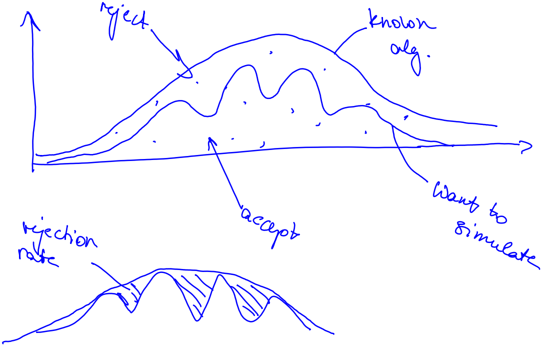
Scale s.t



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- special methods
  - normal distribution
  - Poisson distribution
  - Gamma distribution
  - etc.

Questions for next lecture

While reading *Chapter 4.1–4.2* of [G], ask yourself:

- Which possibilities do you have to sample the same path of a Brownian motion with different accuracy/resolution?
- What is important for sample paths vs. distribution?
- How is the heat equation coupled to Brownian motion? In which sense?
- What is the naive idea of a filtration? How is it related to our daily life?
- Why can't we use "usual" integration for Brownian motion but have to define the Itô integral?
- What are the basic steps for the definition of the Itô integral? What do they tell us?
- In which sense of "uniqueness" should the Itô integral be interpreted?
- What are important properties of the Itô integral?