Exercise 4.1
(Linear transformation of Brownian motion)

i) Let W be a standard d-dimensional Brownian motion and let U be an orthogonal matrix (i.e. $U^T = U^{-1}$). Prove that UW defines a new standard d-dimensional Brownian motion.

Solution: We have that W is a d-dimensional Brownian motion, i.e. it satisfies

1. $W_0 = 0 \in \mathbb{R}^d$,
2. W_t is a.s. continuous,
3. W_t has independent increments, and
4. $W_t - W_s \sim \mathcal{N}(0, t-s)$.

We check if UW satisfies these conditions for an orthogonal matrix U. Since W_0 is the zero vector, obviously UW_0 will also result in the zero vector in $\mathbb{R}^d$. Moreover, neither the a.s. continuity of W_t nor the independence of the increments will be affected by the multiplication of a matrix. The non-trivial thing to check is if the increments remain normally distributed with the same parameters. For this, we utilize the characteristic function to note that

$$\varphi_{UW_t - UW_s}(\xi) = \mathbb{E}\{e^{i\xi^T U(W_t - W_s)}\} = \varphi_{W_t - W_s}(\xi^T U) = e^{-\frac{1}{2}(t-s)\xi^T UU^T \xi} = e^{-\frac{1}{2}(t-s)|\xi|^2}.$$

Here, the first and second equality follows from the definition of the characteristic function, the third from the characteristic function of a Gaussian random variable, and the last from the orthogonality of U.

ii) Let W_1 and W_2 be two independent Brownian motions. For any $\rho \in [-1, 1]$, justify that $\rho W_1 + \sqrt{1-\rho^2} W_2$ and $-\sqrt{1-\rho^2} W_1 + \rho W_2$ are two independent Brownian motions.

Solution: Denote a 2-dimensional Brownian motion by

$$W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}.$$

Note that with

$$U = \begin{bmatrix} \rho & \sqrt{1-\rho^2} \\ -\sqrt{1-\rho^2} & \rho \end{bmatrix}$$

we have that the two given processes are given as the two components of UW. Note that U is orthogonal since

$$UU^T = \begin{bmatrix} \rho & \sqrt{1-\rho^2} \\ -\sqrt{1-\rho^2} & \rho \end{bmatrix} \begin{bmatrix} \rho & -\sqrt{1-\rho^2} \\ \sqrt{1-\rho^2} & \rho \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Using the previously shown statement, we have that both the components are independent Brownian motions.