Exercise 4.2

(Approximation of the integral of a stochastic process)

For a standard Brownian motion, we study the convergence rate of the approximation

$$\Delta I_n := \int_0^1 W_s \, \mathrm{d}s - \frac{1}{n} \sum_{i=0}^{n-1} W_{\frac{i}{n}},$$

as $n \to +\infty$.

i) (Rough estimate). Prove that

$$\mathbb{E}(|\Delta I_n|) \leq \sum_{i=0}^{n-1} \mathbb{E}\left(\int_{\frac{i}{n}}^{\frac{i+1}{n}} \left| W_s - W_{\frac{i}{n}} \right| \mathrm{d}s \right) = \mathcal{O}(n^{-1/2}).$$

<u>Solution:</u> For the first part, we simply note that

$$\begin{aligned}\mathbb{E}(|\Delta I_n|) &= \mathbb{E}\left(\left| \int_0^1 W_s \, \mathrm{d}s - \frac{1}{n} \sum_{i=0}^{n-1} W_{\frac{i}{n}} \right| \right) = \mathbb{E}\left(\left| \sum_{i=0}^{n-1} \int_{\frac{i}{n}}^{\frac{i+1}{n}} W_s \, \mathrm{d}s - \sum_{i=0}^{n-1} \int_{\frac{i}{n}}^{\frac{i+1}{n}} W_{\frac{i}{n}} \, \mathrm{d}s \right| \right) \\ &= \mathbb{E}\left(\left| \sum_{i=0}^{n-1} \int_{\frac{i}{n}}^{\frac{i+1}{n}} W_s - W_{\frac{i}{n}} \, \mathrm{d}s \right| \right) \leq \sum_{i=0}^{n-1} \mathbb{E}\left(\int_{\frac{i}{n}}^{\frac{i+1}{n}} \left| W_s - W_{\frac{i}{n}} \right| \mathrm{d}s \right).\end{aligned}$$

Now, the integrand is the absolute value of a normally distributed random variable, and if we have $X \sim \mathcal{N}(0, \sigma^2)$, then we note that

$$\mathbb{E}(|X|) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} |x| e^{-\frac{1}{2\sigma^2}x^2} \, \mathrm{d}x = \sqrt{\frac{2}{\pi}} \frac{1}{\sigma} \int_0^{\infty} x e^{-\frac{1}{2\sigma^2}x^2} \, \mathrm{d}x = \sqrt{\frac{2}{\pi}} \frac{1}{\sigma} \left[-\sigma^2 e^{-\frac{1}{2\sigma^2}x^2} \right]_0^{\infty} = \sqrt{\frac{2}{\pi}} \sigma.$$

Thus, by putting the expectation inside of the integral (justified by Fubini's theorem) we get that

$$\begin{aligned}\mathbb{E}(|\Delta I_n|) &\leq \sqrt{\frac{2}{\pi}} \sum_{i=0}^{n-1} \int_{\frac{i}{n}}^{\frac{i+1}{n}} \sqrt{s - \frac{i}{n}} \, \mathrm{d}s = \sqrt{\frac{2}{\pi}} \sum_{i=0}^{n-1} \left[\frac{2}{3} (s - i/n)^{3/2} \right]_{\frac{i}{n}}^{\frac{i+1}{n}} \\ &= \sqrt{\frac{8}{9\pi}} \sum_{i=0}^{n-1} n^{-3/2} = \sqrt{\frac{8}{9\pi}} n^{-1/2} = \mathcal{O}(n^{-1/2}),\end{aligned}$$

since $\sigma = \sqrt{s - i/n}$ for $W_s - W_{\frac{i}{n}}$.

ii) Using Lemma A.1.4, prove that ΔI_n is Gaussian distributed. Compute its parameters and conclude that

$$\mathbb{E}(|\Delta I_n|) = \mathcal{O}(n^{-1}).$$

<u>Solution:</u> The lemma states that if we have a sequence $(X_n)_n$ such that $X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$, then it converges in distribution as $n \to +\infty$ if and only if the sequence of the parameters $(\mu_n, \sigma_n^2)_n$ converges. In the case of convergence, it holds that the limit distribution of X_n is Gaussian with parameters $(\lim_n \mu_n, \lim_n \sigma_n)$. Hence, we seek to find a sequence of Gaussian random variables that converges to the limit ΔI_n.

We know from previous task that

$$\Delta I_n = \sum_{i=0}^{n-1} \int_{\frac{i}{n}}^{\frac{i+1}{n}} W_s - W_{\frac{i}{n}} \, \mathrm{d}s.$$

Next, for a given $M \in \mathbb{N}$ we define the partition $\frac{i}{n} =: s_0 < s_1 < \ldots < s_M := \frac{i+1}{n}$. We consider the integral as the limit of a Riemann sum, namely

$$\sum_{j=0}^{M-1} (W_{s_j} - W_{s_0})(s_{j+1} - s_j).$$

Denote $B_j = W_{s_j} - W_{s_0}$, and note that

$$\begin{aligned}
\sum_{j=0}^{M-1} B_j(s_{j+1} - s_j) &= B_{M-1}(s_M - s_{M-1}) + B_{M-2}(s_{M-1} - s_{M-2}) + \ldots + B_0(s_1 - s_0) \\
&= \sum_{j=1}^{M-1} (B_{j-1} - B_j)s_j + B_{M-1}s_M \\
&= \sum_{j=1}^{M-1} (B_{j-1} - B_j)s_j + (B_{M-1} + B_{M-2} - B_{M-2} + \ldots - B_0)s_M \\
&= \sum_{j=1}^{M-1} (B_{j-1} - B_j)s_j + \sum_{j=1}^{M-1} (B_j - B_{j-1})s_M \\
&= \sum_{j=1}^{M-1} (B_j - B_{j-1})(s_M - s_j) =: S_M.
\end{aligned}$$

Note here that S_M is a sum of independent Gaussian variables, and hence is itself a Gaussian variable. Its expectation obviously satisfies $\mathbb{E}\{S_M\} = 0$ by linearity. Since they each are independent, the variance is found as

$$\begin{aligned}
\mathrm{Var}(S_M) = \{B_j \text{ independent}\} &= \sum_{j=1}^{M-1} \mathrm{Var}((B_j - B_{j-1})(s_M - s_j)) \\
= \{\mathrm{Var}(cX) = c^2\mathrm{Var}(X)\} &= \sum_{j=1}^{M-1} (s_M - s_j)^2(s_j - s_{j-1}) =: \sigma_M^2.
\end{aligned}$$

We have thus found a sequence $\{S_M\}_M$ of Gaussian random variables with parameters $(0, \sigma_M^2)$. In the limit we have $\mu = 0$ and

$$\lim_{M\to\infty} \sigma_M^2 = \int_{\frac{i}{n}}^{\frac{i+1}{n}} \Big(\frac{i+1}{n} - s\Big)^2 \mathrm{d}s = -\frac{1}{3}\left[\Big(\frac{i+1}{n} - s\Big)^3\right]_{\frac{i}{n}}^{\frac{i+1}{n}} = \frac{1}{3n^3}.$$

Hence, by applying the given lemma, we get in the limit that

$$\int_{\frac{i}{n}}^{\frac{i+1}{n}} W_s - W_{\frac{i}{n}} \, \mathrm{d}s \sim \mathcal{N}\Big(0, \frac{1}{3n^3}\Big).$$

The integrals are all independent, so when we apply the sum, the total variance will just be summed, and we get that

$$\Delta I_n \sim \mathcal{N}\Big(0, \frac{1}{3n^2}\Big),$$

Hence we find (by considering the expectation of the absolute value of a Gaussian random varaible)

$$\mathbb{E}(|\Delta I_n|) = \sqrt{\frac{2}{\pi}}\sigma = \sqrt{\frac{2}{3\pi}}\frac{1}{n} = \mathcal{O}(n^{-1}).$$

iii) A more generic proof of the above estimate consists of writing

$$\Delta I_n := \sum_{i=0}^{n-1} \int_{\frac{i}{n}}^{\frac{i+1}{n}} \Big(\frac{i+1}{n} - s\Big)\,\mathrm{d}W_s.$$

Show this by applying the Itô formula to $s \mapsto (\frac{i+1}{n} - s)(W_s - W_{\frac{i}{n}})$ on each interval $[\frac{i}{n}, \frac{i+1}{n}]$. Using the Itô isometry, derive $\mathbb{E}(|\Delta I_n|^2) = \mathcal{O}(n^{-2})$ and therefore the announced estimate.

<u>Solution:</u> Let

$$\begin{aligned} f(t, W_t) &= \Big(\frac{i+1}{n} - t\Big)\Big(W_t - W_{\frac{i}{n}}\Big), \\ \frac{\partial}{\partial t} f(t, W_t) &= -(W_s - W_{\frac{i}{n}}), \\ \nabla_x f(t, W_t) &= \Big(\frac{i+1}{n} - s\Big). \end{aligned}$$

Applying the Itô formula, we get

$$f(t, W_t) = f\Big(\frac{i}{n}, W_{\frac{i}{n}}\Big) + \int_{\frac{i}{n}}^{t} -(W_s - W_{\frac{i}{n}})\,\mathrm{d}s + \int_{\frac{i}{n}}^{t} \Big(\frac{i+1}{n} - s\Big)\,\mathrm{d}W_s.$$

Now let $t = \frac{i+1}{n}$. The two f-terms vanish and by passing the standard integral to the left hand side we get

$$\int_{\frac{i}{n}}^{\frac{i+1}{n}} (W_s - W_{\frac{i}{n}})\,\mathrm{d}s = \int_{\frac{i}{n}}^{\frac{i+1}{n}} \Big(\frac{i+1}{n} - s\Big)\,\mathrm{d}W_s.$$

The last part of the task follows as

$$
\begin{aligned}
\mathbb{E}\{|\Delta I_n|^2\} &= \mathbb{E}\left(\left|\sum_{i=0}^{n-1}\int_{\frac{i}{n}}^{\frac{i+1}{n}}\Big(\frac{i+1}{n}-s\Big)\,\mathrm{d}W_s\right|^2\right)\\
&= \mathbb{E}\left(\left(\sum_{i=0}^{n-1}\int_{\frac{i}{n}}^{\frac{i+1}{n}}\Big(\frac{i+1}{n}-s\Big)\,\mathrm{d}W_s\right)\cdot\left(\sum_{j=0}^{n-1}\int_{\frac{j}{n}}^{\frac{j+1}{n}}\Big(\frac{j+1}{n}-s\Big)\,\mathrm{d}W_s\right)\right)\\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\mathbb{E}\left(\left(\int_{\frac{i}{n}}^{\frac{i+1}{n}}\Big(\frac{i+1}{n}-s\Big)\,\mathrm{d}W_s\right)\cdot\left(\int_{\frac{j}{n}}^{\frac{j+1}{n}}\Big(\frac{j+1}{n}-s\Big)\,\mathrm{d}W_s\right)\right)\\
&= \sum_{i=0}^{n-1}\mathbb{E}\left(\left(\int_{\frac{i}{n}}^{\frac{i+1}{n}}\Big(\frac{i+1}{n}-s\Big)\,\mathrm{d}W_s\right)^2\right)\\
&= \sum_{i=0}^{n-1}\mathbb{E}\left(\int_{\frac{i}{n}}^{\frac{i+1}{n}}\Big(\frac{i+1}{n}-s\Big)^2\,\mathrm{d}s\right)\\
&= \sum_{i=0}^{n-1}\left[-\frac{1}{3}\Big(\frac{i+1}{n}-s\Big)^3\right]_{\frac{i}{n}}^{\frac{i+1}{n}}\\
&= n\frac{1}{3n^3} = \mathcal{O}(n^{-2}).
\end{aligned}
$$

In the fourth equality we used the fact that each mixed term of the double sum will result in two independent Gaussian random variables, both centered, and hence they become zero. Moreover, the Itô isometry is used in the fifth equality.

iv) Proceeding as in *(iii)*, extend the previous estimate to

$$\Delta I'_n := \int_0^1 X_s\,\mathrm{d}s - \frac{1}{n}\sum_{i=0}^{n-1} X_{\frac{i}{n}},$$

where X is a scalar Itô process with bounded coefficients.

<u>Solution:</u> As earlier, we can rewrite it as

$$\Delta I'_n = \sum_{i=0}^{n-1}\int_{\frac{i}{n}}^{\frac{i+1}{n}} X_s - X_{\frac{i}{n}}\,\mathrm{d}s.$$

Now, consider the function $f(s,X_s) = (\frac{i+1}{n}-s)(X_s - X_{\frac{i}{n}})$, and note that

$$
\begin{aligned}
\partial_t f(t,X_t) &= -(X_t - X_{\frac{i}{n}}),\\
\nabla_x f(t,X_t) &= \frac{i+1}{n} - t.
\end{aligned}
$$

Applying Itô's formula, we get

$$\Big(\frac{i+1}{n}-t\Big)(X_t - X_{\frac{i}{n}}) = \int_{\frac{i}{n}}^{t} -(X_s - X_{\frac{i}{n}}) + b_s\Big(\frac{i+1}{n}-s\Big)\,\mathrm{d}s - \int_{\frac{i}{n}}^{t}\sigma_s\Big(\frac{i+1}{n}-s\Big)\,\mathrm{d}W_s.$$

Let $t = \frac{i+1}{n}$ and we find that

$$\begin{aligned}\int_{\frac{i}{n}}^{\frac{i+1}{n}} X_s - X_{\frac{i}{n}}\,\mathrm{d}s &= \int_{\frac{i}{n}}^{\frac{i+1}{n}} b_s\Big(\frac{i+1}{n} - s\Big)\,\mathrm{d}s + \int_{\frac{i}{n}}^{\frac{i+1}{n}} \sigma_s\Big(\frac{i+1}{n} - s\Big)\,\mathrm{d}W_s\\ &\le b\frac{1}{2n^2} + \sigma\int_{\frac{i}{n}}^{\frac{i+1}{n}}\Big(\frac{i+1}{n} - s\Big)\,\mathrm{d}W_s,\end{aligned}$$

where $b = \max_s b_s$ and $\sigma = \max_s \sigma_s$. Given this, we get

$$\begin{aligned}\mathbb{E}\{|\Delta I_n'|^2\} &\le \mathbb{E}\left\{\left(b\frac{1}{2n} + \sum_{i=0}^{n-1}\sigma\int_{\frac{i}{n}}^{\frac{i+1}{n}}\Big(\frac{i+1}{n} - s\Big)\,\mathrm{d}W_s\right)^2\right\}\\ &= \mathbb{E}\left\{\frac{b^2}{4n^2} + \frac{b\sigma}{n}\sum_{i=0}^{n-1}\int_{\frac{i}{n}}^{\frac{i+1}{n}}\Big(\frac{i+1}{n} - s\Big)\,\mathrm{d}W_s + \sigma^2\left(\sum_{i=0}^{n-1}\int_{\frac{i}{n}}^{\frac{i+1}{n}}\Big(\frac{i+1}{n} - s\Big)\,\mathrm{d}W_s\right)^2\right\}\\ &= \frac{b^2}{4n^2} + \frac{b\sigma}{n}\sum_{i=0}^{n-1}\mathbb{E}\left\{\int_{\frac{i}{n}}^{\frac{i+1}{n}}\Big(\frac{i+1}{n} - s\Big)\,\mathrm{d}W_s\right\} + \sigma^2\mathbb{E}\Bigg(\underbrace{\left(\sum_{i=0}^{n-1}\int_{\frac{i}{n}}^{\frac{i+1}{n}}\Big(\frac{i+1}{n} - s\Big)\,\mathrm{d}W_s\right)^2}_{=|\Delta I_n|^2}\Bigg)\\ &= \frac{b^2}{4n^2} + 0 + \frac{\sigma^2}{3n^2}\\ &= \frac{3b^2 + 4\sigma^2}{12n^2} = \mathcal{O}(n^{-2}),\end{aligned}$$

where we have used the fact that the stochastic integral is centered, as well as the result from previous task.