Exercise 4.3
(Approximation of stochastic integral)

We consider the convergence rate of the approximation

$$\Delta J_n := \int_0^1 Z_s \, \mathrm{d}W_s - \sum_{i=0}^{n-1} Z_{\frac{i}{n}} (W_{\frac{i+1}{n}} - W_{\frac{i}{n}})$$

where $Z_s := f(s, W_s)$ for some function f, such that $\mathbb{E} \int_0^1 |Z_s|^2 \, \mathrm{d}s + \sup_{i<n} \mathbb{E}(|Z_{\frac{i}{n}}|^2) < +\infty$. We illustrate that the convergence order is, under mild conditions, equal to $1/2$ but it can be smaller for irregular f.

i) Show that

$$\mathbb{E}(|\Delta J_n|^2) = \mathbb{E}\Big(\sum_{i=0}^{n-1} \int_{\frac{i}{n}}^{\frac{i+1}{n}} |Z_s - Z_{\frac{i}{n}}|^2 \, \mathrm{d}s \Big).$$

<u>Solution:</u> We see this by first noting that

$$\sum_{i=0}^{n-1} Z_{\frac{i}{n}} (W_{\frac{i+1}{n}} - W_{\frac{i}{n}}) = \sum_{i=0}^{n-1} \int_{\frac{i}{n}}^{\frac{i+1}{n}} Z_{\frac{i}{n}} \, \mathrm{d}W_s = \int_0^1 \sum_{i=0}^{n-1} Z_{\frac{i}{n}} \mathbb{1}_{\{\frac{i}{n}, \frac{i+1}{n}\}} \, \mathrm{d}W_s.$$

Thus, replacing this expression in ΔJ_n and applying Itô isometry, we get

$$\begin{aligned}
\mathbb{E}(|\Delta J_n|^2) &= \mathbb{E}\left(\left(\int_0^1 Z_s \, \mathrm{d}W_s - \sum_{i=0}^{n-1} Z_{\frac{i}{n}} (W_{\frac{i+1}{n}} - W_{\frac{i}{n}}) \right)^2 \right) \\
&= \mathbb{E}\left(\left(\int_0^1 \Big(Z_s - \sum_{i=0}^{n-1} Z_{\frac{i}{n}} \mathbb{1}_{\{\frac{i}{n}, \frac{i+1}{n}\}} \Big) \, \mathrm{d}W_s \right)^2 \right) \\
&= \mathbb{E}\left(\int_0^1 \Big(Z_s - \sum_{i=0}^{n-1} Z_{\frac{i}{n}} \mathbb{1}_{\{\frac{i}{n}, \frac{i+1}{n}\}} \Big)^2 \, \mathrm{d}s \right) \\
&= \mathbb{E}\left(\sum_{i=0}^{n-1} \int_{\frac{i}{n}}^{\frac{i+1}{n}} \Big(Z_s - Z_{\frac{i}{n}} \Big)^2 \, \mathrm{d}s \right).
\end{aligned}$$

ii) When $Z_s = W_s$, show that $\mathbb{E}(|\Delta J_n|^2) \sim \mathrm{Cst}\, n^{-1}$ for some positive constant.

<u>Solution:</u> This is seen as

$$\begin{aligned}
\mathbb{E}(|\Delta J_n|^2) &= \sum_{i=0}^{n-1} \mathbb{E}\left(\int_{\frac{i}{n}}^{\frac{i+1}{n}} |W_s - W_{\frac{i}{n}}|^2 \, \mathrm{d}s \right) = \sum_{i=0}^{n-1} \int_{\frac{i}{n}}^{\frac{i+1}{n}} \mathbb{E}\big(|\underbrace{W_s - W_{\frac{i}{n}}}_{\sim \mathcal{N}(0, s - \frac{i}{n})}|^2 \big) \, \mathrm{d}s \\
&= \sum_{i=0}^{n-1} \int_{\frac{i}{n}}^{\frac{i+1}{n}} \Big(s - \frac{i}{n} \Big) \, \mathrm{d}s = \sum_{i=0}^{n-1} \left[\frac{1}{2} \Big(s - \frac{i}{n} \Big)^2 \right]_{\frac{i}{n}}^{\frac{i+1}{n}} = \sum_{i=0}^{n-1} \frac{1}{2n^2} \\
&= \frac{1}{2n}.
\end{aligned}$$

Here we first used the result from *(i)*, then Fubini to move the expectation inside the integral, followed by noting that we have the variance as integrand (since $\mathbb{V}\mathrm{ar}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$ and we have zero expectation from the increment).

iii) Assuming that f is bounded, smooth with bounded derivatives, prove that $\mathbb{E}(|\Delta J_n|^2) = \mathcal{O}(n^{-1})$.

<u>Solution:</u> Since we are given the condition on smooth bounded derivatives, we are encouraged to apply Itô's formula on $Z_s := f(s, W_s)$. Applying the formula on the interval $[\frac{i}{n}, s]$ gives

$$\begin{aligned}
Z_s - Z_{\frac{i}{n}} &= \int_{\frac{i}{n}}^{s} \partial_t f(s, W_s) + b(s, W_s)\nabla_x f(s, W_s) + \frac{1}{2}\sum_{k,l=1}^{q} (\ldots)\,\mathrm{d}s + \int_{\frac{i}{n}}^{s} \sigma(s, W_s)\nabla_x f(s, W_s)\,\mathrm{d}W_s \\
&\leq C_1\Big(s - \frac{i}{n}\Big) + C_2(W_s - W_{\frac{i}{n}}),
\end{aligned}$$

since everything is nice and bounded. Hence, by inserting this into the expression ΔJ_n and applying Fubini to move the expectation into the integral, we find that

$$\begin{aligned}
\mathbb{E}(|\Delta J_n|^2) &= \mathbb{E}\left(\sum_{i=0}^{n-1}\int_{\frac{i}{n}}^{\frac{i+1}{n}} |Z_s - Z_{\frac{i}{n}}|^2\,\mathrm{d}s\right) \\
&\leq \sum_{i=0}^{n-1}\int_{\frac{i}{n}}^{\frac{i+1}{n}} \left(C_1^2\Big(s - \frac{i}{n}\Big)^2 + 2C_1C_2\Big(s - \frac{i}{n}\Big)\mathbb{E}(W_s - W_{\frac{i}{n}}) + C_2^2\mathbb{E}(|W_s - W_{\frac{i}{n}}|^2)\right)\mathrm{d}s \\
&= \sum_{i=0}^{n-1}\Big(\frac{C_1^2}{3n^3} + \frac{C_2^2}{2n^2}\Big) = \mathcal{O}(n^{-1}),
\end{aligned}$$

where the mixed term vanished due to the zero expectation of the increment of Brownian motion.

iv) Assume that Z is a square-integrable martingale. Show that $\mathbb{E}(|Z_s - Z_{\frac{i}{n}}|^2) \leq \mathbb{E}(|Z_{\frac{i+1}{n}}|^2) - \mathbb{E}(|Z_{\frac{i}{n}}|^2)$, and thus $\mathbb{E}(|\Delta J_n|^2) \leq (\mathbb{E}(|Z_1|^2) - \mathbb{E}(|Z_0|^2)n^{-1}$.

<u>Solution:</u> First of all, we note that for $s < t$ we have

$$\mathbb{E}(Z_t Z_s|\mathcal{F}_s) = Z_s\mathbb{E}(Z_t|\mathcal{F}_s) = Z_s^2,$$

since Z_s is $\mathcal{F}_s$-measurable. Thus, by applying the towering property with a filtration $\mathcal{F}_{\frac{i}{n}}$ we find that

$$\begin{aligned}
\mathbb{E}(|Z_s - Z_{\frac{i}{n}}|^2) &= \mathbb{E}(\mathbb{E}((Z_s - Z_{\frac{i}{n}})^2|\mathcal{F}_{\frac{i}{n}})) = \mathbb{E}(\mathbb{E}(Z_s^2 - 2Z_sZ_{\frac{i}{n}} + Z_{\frac{i}{n}}^2|\mathcal{F}_{\frac{i}{n}})) \\
&= \mathbb{E}(\mathbb{E}(Z_s^2 + Z_{\frac{i}{n}}^2|\mathcal{F}_{\frac{i}{n}}) - \mathbb{E}(2Z_sZ_{\frac{i}{n}}|\mathcal{F}_{\frac{i}{n}})) = \mathbb{E}(\mathbb{E}(Z_s^2 - Z_{\frac{i}{n}}^2|\mathcal{F}_{\frac{i}{n}})) \\
&= \mathbb{E}(Z_s^2 - Z_{\frac{i}{n}}^2) = \mathbb{E}(Z_s^2) - \mathbb{E}(Z_{\frac{i}{n}}^2),
\end{aligned}$$

by linearity of the conditional expectation. Moreover, note that this result implies that

$$0 \leq \mathbb{E}(|Z_{\frac{i+1}{n}} - Z_s|^2) = \mathbb{E}(Z_{\frac{i+1}{n}}^2) - \mathbb{E}(Z_s^2),$$

so $\mathbb{E}(Z_s^2) \leq \mathbb{E}(Z_{\frac{i+1}{n}}^2)$, and thus we find that

$$\mathbb{E}(|Z_s - Z_{\frac{i}{n}}|^2) = \mathbb{E}(Z_s^2) - \mathbb{E}(Z_{\frac{i}{n}}^2) \leq \mathbb{E}(Z_{\frac{i+1}{n}}^2) - \mathbb{E}(Z_{\frac{i}{n}}^2).$$

An easy consequence of this is

$$\begin{aligned}\mathbb{E}(|\Delta J_n|^2) &= \sum_{i=0}^{n-1} \int_{\frac{i}{n}}^{\frac{i+1}{n}} \mathbb{E}(|Z_s - Z_{\frac{i+1}{n}}|^2)\,\mathrm{d}s \le \sum_{i=0}^{n-1} \int_{\frac{i}{n}}^{\frac{i+1}{n}} \mathbb{E}(|Z_{\frac{i+1}{n}}|^2) - \mathbb{E}(|Z_{\frac{i}{n}}|^2)\,\mathrm{d}s \\ &= \sum_{i=0}^{n-1} (\mathbb{E}(|Z_{\frac{i+1}{n}}|^2) - \mathbb{E}(|Z_{\frac{i}{n}}|^2))\frac{1}{n} = (\mathbb{E}(|Z_1|^2) - \mathbb{E}(|Z_0|^2))n^{-1}.\end{aligned}$$