Exercise 4.4

(Exact simulation of Ornstein–Uhlenbeck process)

Let us consider the Ornstein–Uhlenbeck process $(X_t)_{t\geq 0}$, the solution of

$$X_t = x_0 - a\int_0^t X_s \,\mathrm{d}s + \sigma W_t,$$

where $x_0 \in \mathbb{R}$, $\sigma > 0$, and $(W_t)_{t\geq 0}$ is a standard Brownian motion.

i) By applying the Itô formula to $e^{at}X_t$, give an explicit representation for X_t in terms of stochastic integrals.

Solution: We have $f(t, X_t) = e^{at}X_t$, to which we compute the derivatives as

$$\partial_t f(t, X_t) = ae^{at}X_t, \quad \text{and} \quad \nabla_x f(t, X_t) = e^{at},$$

and the second spatial derivative vanishes. Applying Itô's formula yields

$$\begin{aligned} e^{at}X_t &= f(0, X_0) + \int_0^t ae^{as}X_s + b(s, X_s)e^{as}\,\mathrm{d}s + \int_0^t e^{as}\sigma(s, X_s)\,\mathrm{d}W_s \\ &= x_0 + \int_0^t ae^{as}X_s - aX_s e^{as}\,\mathrm{d}s + \sigma\int_0^t e^{as}\,\mathrm{d}W_s. \end{aligned}$$

Multiply each side by e^{-at} to get

$$X_t = x_0e^{-at} + \sigma\int_0^t e^{a(s-t)}\,\mathrm{d}W_s.$$

ii) Deduce the explicit distribution of $(X_{t_1}, \ldots, X_{t_n})$.

Solution: Since the first term on the right hand side is deterministic, and the stochastic integral is Gaussian (since $e^{a(t-s)}$ is deterministic), we have that they are Gaussian. To determine their parameters, we first note that

$$\mathbb{E}(X_t) = x_0e^{-at},$$

since the stochastic integral is centered. To find the covariance, we first note that

$$\begin{aligned} \mathbb{E}\{X_tX_s\} &= \mathbb{E}\left\{\left(x_0e^{-at} + \sigma\int_0^t e^{a(t'-t)}\,\mathrm{d}W_{t'}\right)\left(x_0e^{-as} + \sigma\int_0^s e^{a(t'-s)}\,\mathrm{d}W_{t'}\right)\right\} \\ &= x_0^2e^{-2a(t+s)} + 0 + 0 + \mathbb{E}\left\{\sigma^2\int_0^t e^{a(t'-t)}\,\mathrm{d}W_{t'}\int_0^s e^{a(t'-s)}\,\mathrm{d}W_{t'}\right\}, \end{aligned}$$

since the stochastic integrals are centered. Hence we get (assuming $s \leq t$)

$$
\begin{aligned}
\operatorname{Cov}\{X_t, X_s\} &= \mathbb{E}\{X_t X_s\} - \mathbb{E}\{X_t\}\mathbb{E}\{X_s\} \\
&= \mathbb{E}\left\{\sigma^2 \int_0^t e^{a(t'-t)} \,\mathrm{d}W_{t'} \int_0^s e^{a(t'-s)} \,\mathrm{d}W_{t'}\right\} \\
&= \mathbb{E}\left\{\sigma^2 \int_0^t e^{a(t'-t)} \,\mathrm{d}W_{t'} \int_0^t e^{a(t'-s)} \mathbb{1}_{\{0,s\}} \,\mathrm{d}W_{t'}\right\} \\
&= \sigma^2 \mathbb{E}\left\{\int_0^t e^{a(t'-t)} e^{a(t'-s)} \mathbb{1}_{\{0,s\}} \,\mathrm{d}t'\right\} \\
&= \sigma^2 e^{-a(t+s)} \int_0^s e^{2at'} \,\mathrm{d}t' \\
&= \frac{\sigma^2}{2a} e^{-a(t+s)} (e^{2as} - 1) \\
&= \frac{\sigma^2}{2a} e^{-a(t-s)} (1 - e^{-2as}),
\end{aligned}
$$

where in the second line, the fact that the first and last term cancels out has been used, and in the third line we add an indicator function so that we can apply the covariance property (stated on page 136 in the course literature).

iii) Find two functions $\alpha(t)$ and $\beta(t)$ such that $(X_t)_{t\geq 0}$ has the same distribution as $(Y_t)_{t\geq 0}$ with $Y_t = \alpha(t)(x_0 + W_{\beta(t)})$.

<u>Solution:</u> Obviously Y_t is Gaussian, since Brownian motion is so. To have the same mean as X_t, we require that $\alpha(t) = e^{-at}$ (since Brownian motion has mean zero). For the covariance, we note that

$$\mathbb{E}\{Y_t\}\mathbb{E}\{Y_s\} = x_0^2 e^{-a(t+s)},$$

and that

$$
\begin{aligned}
\mathbb{E}\{Y_t Y_s\} &= x_0^2 e^{-a(t+s)} + 0 + 0 + e^{-a(t+s)} \mathbb{E}\{W_{\beta(t)} W_{\beta(s)}\} \\
&= x_0^2 e^{-a(t+s)} + e^{-a(t+s)} \min\{\beta(t), \beta(s)\},
\end{aligned}
$$

since Brownian motion has mean zero. To get the same covariance as for X_t, we need $\beta(t)$ to satisfy

$$e^{-a(t+s)} \min\{\beta(t), \beta(s)\} = \frac{\sigma^2}{2a} e^{-a(t-s)} (1 - e^{-2as}),$$

which can be rewritten as

$$\beta(s) = \frac{\sigma^2}{2a}(e^{2as} - 1).$$

In conclusion, we have that

$$Y_t = e^{-at}\big(x_0 + W_{\frac{\sigma^2}{2a}(e^{2at}-1)}\big)$$

has the same distribution as X_t.