Exercise 4.5
(Transformations of SDE and PDE)

For any $t \in [0,T)$ and $x \in \mathbb{R}$, we denote by $(X_s^{t,x}, s \in [t,T])$ the solution to

$$X_s = x + \int_t^s b(X_r)\,\mathrm{d}r + \int_t^s \sigma(X_r)\,\mathrm{d}W_r, \qquad t \le s \le T,$$

where the coefficients $b, \sigma : \mathbb{R} \to \mathbb{R}$ are smooth with bounded derivatives, and $\sigma(x) \ge c > 0$. For a given Borel set $A \subset \mathbb{R}$ we define $u(t,x) := \mathbb{P}(X_T^{t,x} \in A)$. We assume in the following $u(t,x) > 0$ for any $(t,x) \in [0,T) \times \mathbb{R}$, and that appropriate smoothness assumptions are satisfied (namely, $u \in \mathcal{C}^{1,2}([0,T) \times \mathbb{R})$).

i) Let $x_0 \in \mathbb{R}$ and f be a bounded continuous function. Using the PDE satisfied by u on $[0,T) \times \mathbb{R}$, show that

$$\mathbb{E}(f(X_t)|X_T \in A) = \frac{\mathbb{E}(f(X_t)u(t,X_t))}{u(0,x_0)}, \qquad \forall t < T,$$

where $X_t = X_t^{0,x_0}$ to simplify.

<u>Solution:</u> Using the definition of conditional expectation, and applying the towering property with a filtration $\mathcal{F}_t$, we find that

$$\begin{aligned}\mathbb{E}(f(X_t)|X_T \in A) &= \frac{\mathbb{E}(f(X_t)\mathbb{1}_{X_T \in A})}{\mathbb{P}(X_T \in A)} = \frac{\mathbb{E}(\mathbb{E}(f(X_t)\mathbb{1}_{X_T\in A}|\mathcal{F}_t))}{u(0,x_0)} \\ &= \frac{\mathbb{E}(f(X_t)\mathbb{E}(\mathbb{1}_{X_T\in A}|\mathcal{F}_t))}{u(0,x_0)} = \frac{\mathbb{E}(f(X_t)\mathbb{P}(X_T^{t,X_t} \in A))}{u(0,x_0)} \\ &= \frac{\mathbb{E}(f(X_t)u(t,X_t))}{u(0,x_0)}.\end{aligned}$$

ii) We assume that for any $s \le t < T$ the equation

$$\overline{X}_r = x + \int_s^r \left(b(\overline{X}_w) + \sigma^2(\overline{X}_w)\frac{\partial_x u}{u}(w, \overline{X}_w)\right)\mathrm{d}w + \int_s^r \sigma(\overline{X}_w)\,\mathrm{d}W_w, \quad s \le r \le t$$

has a unique solution, denoted by $(\overline{X}_r^{s,x}, s \le r \le t)$. We set $v_t(s,x) := \mathbb{E}\big(f(\overline{X}_t^{s,x})\big)$.

a) What is the PDE solved by $(s,x) \mapsto v_t(s,x)$ on $[0,t) \times \mathbb{R}$?

b) Applying the Itô formula to $u(s,X_s)$ and $v_t(s,x)$, $0 \le s \le t$, and then to $u(s,X_s)v_t(s,X_s)$, show

$$\mathbb{E}(f(X_t)u(t,X_t)) = v_t(0,x_0)u(0,x_0), \quad \forall t < T.$$

c) Conclude that for any $t < T$, the distribution of X_t given $\{X_T \in A\}$ is the distribution of $\overline{X}_t^{0,x_0}$.

Solution:

a) The Feynman–Kac formula states that the solution to

$$\begin{cases} \partial_t u(t,x) + \mathcal{L}u(t,x) - k(t,x)u(t,x) + g(t,x) = 0, & t < T,\ x \in \mathbb{R}^d, \\ u(T,x) = f(x), & x \in \mathbb{R}^d, \end{cases}$$

is given by

$$u(t,x) = \mathbb{E}\left(f(X_T^{t,x})e^{-\int_t^T k(r,X_r^{t,x})\,\mathrm{d}r} + \int_t^T g(s,X_s^{t,x})e^{-\int_t^s k(r,X_r^{t,x})\,\mathrm{d}r}\,\mathrm{d}s \right).$$

We consider $v_t(s,x) := \mathbb{E}\big(f(\overline{X}_t^{s,x})\big)$, i.e. above $k = 0$ and $g = 0$, and hence the corresponding equation is

$$\begin{cases} \partial_s v_t(s,x) + \mathcal{L}v_t(s,x) = 0, & t < T,\ x \in \mathbb{R}^d, \\ u(T,x) = f(x), & x \in \mathbb{R}^d, \end{cases}$$

where the operator is defined by

$$\mathcal{L} = \frac{\sigma(x)^2}{2}\partial_{xx} + \left(b(x) + \sigma(x)^2\frac{\partial_x u(s,x)}{u(s,x)}\right)\partial_x.$$

b) At first, we note that Itô's formula on $v_t(s,x)u(s,x)$ gives

$$\mathrm{d}(v_t(s,X_s)u(s,X_s)) = v_t(s,X_s)\mathrm{d}(u(s,X_s)) + u(s,X_s)\mathrm{d}(v_t(s,X_s)) + \mathrm{d}\langle v_t(s,X_s), u(s,X_s)\rangle. \quad (1)$$

To find an expression for $\mathrm{d}v_t$, we note that Itô's formula on $f(\overline{X}_t)$ is

$$\mathrm{d}f(\overline{X}_t) = (\partial_t + \mathcal{L})f(\overline{X}_t)\mathrm{d}t + \{\text{Diffusion}\}\mathrm{d}W_t.$$

When taking the expected value of above expression, the diffusion term vanishes and we find that

$$\mathrm{d}v_t = (\partial_t + \mathcal{L})v_t\mathrm{d}t.$$

Next, we want to see what happens to the $\mathrm{d}u$-term. Note that $u(s,X_s)$ is a martingale, since for any $r < s$ we have that

$$\mathbb{E}(u(s,X_s)|\mathcal{F}_r) = \mathbb{E}(\mathbb{E}(\mathbb{1}_{X_T \in A}|\mathcal{F}_s)|\mathcal{F}_r) = \mathbb{E}(\mathbb{1}_{X_T \in A}|\mathcal{F}_r) = u(r,X_r),$$

where in the first step we used the definition of $u(t,x)$, and in the second step the fact that we "only know" $\mathcal{F}_r$ which cancels the additional information of $\mathcal{F}_s$. Since $u(s,X_s)$ is a martingale, it holds that

$$\mathbb{E}\left(\int_0^r v_t(s,X_s)\,\mathrm{d}(u(s,X_s))\right) = 0,$$

which can be seen by analyzing its corresponding Riemann sum. Moreover, the covariance term vanishes since

$$\mathrm{d}\langle v_t, u\rangle = \ldots\mathrm{d}t^2 + \ldots\mathrm{d}t\mathrm{d}W_t + 0\mathrm{d}W_t^2,$$

since v_t has no diffusion term. Hence, integrating (1) and taking the expectation yields

$$\begin{aligned}\mathbb{E}\left(v_t(r, X_r)u(r, X_r)\right) &= \mathbb{E}(v_t(0, x_0)u(0, x_0)) + \mathbb{E}\left(\int_0^r u(r, X_r)(\partial_t v_t(s, X_s) + \mathcal{L}u(s, X_s))\,\mathrm{d}t\right) \\ &= v_t(0, x_0)u(0, x_0),\end{aligned}$$

since the integrand is equal to zero due to the PDE v_t satisfies.

c) Using the previous tasks, we find that

$$\mathbb{E}(f(X_t)|X_T \in A) = \frac{\mathbb{E}(f(X_t)u(t, X_t))}{u(0, x_0)} = \frac{v_t(0, x_0)u(0, x_0)}{u(0, x_0)} = v_t(0, x_0) = \mathbb{E}(f(\overline{X}_t^{0,x_0})).$$

Since this holds for any continuous bounded function f, the two distributions coincide.

iii) In the case $b = 0$, $\sigma(x) = 1$ and $A = (y - R, y + R)$, show that $\frac{\partial_x u(t,x)}{u(t,x)} \to -\frac{x-y}{T-t}$ for any (t, x) as $R \to 0$. Interpret the solution to the following equation in terms of a Brownian bridge:

$$\overline{X}_t = x_0 - \int_0^t \frac{\overline{X}_s - y}{T - s}\,\mathrm{d}s + W_t.$$

<u>Solution:</u> Given the coefficients, we have that

$$X_T^{t,x} = x + W_T - W_t \sim \mathcal{N}(0, T - t).$$

Given $Y \sim \mathcal{N}(0, T - t)$ we have the c.d.f $\Phi(x) = \mathbb{P}(Y \leq x)$. We find that

$$\begin{aligned}\frac{\partial_x u(t, x)}{u(t, x)} &= \frac{\partial_x(\mathbb{P}(Y \leq y - x + R) - \mathbb{P}(Y \leq y - x - R))}{\mathbb{P}(Y \leq y - x + R) - \mathbb{P}(Y \leq y - x - R)} \\ &= \frac{\Phi'(y - x + R) - \Phi'(y - x + R)}{\Phi(y - x + R) - \Phi(y - x + R)} \xrightarrow[R\to 0]{} \frac{\Phi''(y - x)}{\Phi'(y - x)}.\end{aligned}$$

Now we have that $\Phi'(x)$ is a Gaussian pdf, so we can insert the formulas

$$\begin{aligned}\Phi'(x) &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}, \\ \Phi''(x) &= \frac{1}{\sigma\sqrt{2\pi}}\left(-\frac{x}{\sigma^2}\right) e^{-\frac{x^2}{2\sigma^2}},\end{aligned}$$

and hence get that

$$\frac{\Phi''(y - x)}{\Phi'(y - x)} = -\frac{y - x}{T - t}.$$

Since we showed that the conditional expectation coincide, and due to the convergence we just showed, we find that the solution $\overline{X}_t$ of the SDE can be interpreted as the distribution of the Brownian bridge starting at x_0 and reaching y at time T.